Andrew Young School of Policy Studies
Research Paper Series

Working Paper 06-09
August 2005

Department of Economics

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August 2005

*Helpful comments by Luc Bovens, Tania Burchardt, Udo Ebert, Amartya Sen and by members of the Human Development Report Office at the UN as well as by seminar participants at the universities of Heidelberg and Oslo, LSE and Queen Mary College London are gratefully acknowledged. We also wish to thank Eugenio Peluso and Ernesto Savaglio for allowing us to study some of their unpublished work. Finally, we are grateful to Christian Aumann for preparing the figures of this paper.

Abstract. Human development is about expanding the choices human beings have to lead lives that they value and is captured by its capability sets which consist of various functioning vectors. The standard of living is then reflected in capability sets. This paper proposes some particular ways of measuring the standard of living available either to an individual or a whole country, when the direction of the development of society represented by a reference functioning vector or a reference cone is uncertain. We provide axiomatic characterizations of the various measures proposed.

Keywords: Functionings, Capabilities, Uncertainty, Social Progress, Standard of Living

JEL Classification Numbers: C78, D63, D71, D74, O12

1 Introduction

Income and wealth are important factors in order to provide and secure a decent standard of living. Economic growth may help to improve this situation. But human development is about much more. As the human development report 2001 asserts, development is about expanding the choices human beings have to lead lives that they value. Fundamental is "building human capabilities – the range of things that people can do or be in life" (Human Development Report (HDR), p. 9). And the report spells out the most basic capabilities for human development: To lead a long and healthy life, to be knowledgeable, and to have access to the resources needed for a decent standard of living.

The human development index aggregates these three basic dimensions of human development into one numerical index, a summary measure. This reduction procedure involves an exercise in weighting. Clearly, a change of weights means affecting the aggregate outcome. Anand and Sen (1997) admit that "there is an inescapable arbitrariness " (p. 16) in this exercise. Earlier on in their paper, they are more explicit on this issue. "Since any choice of weights should be open to questioning and debating in public discussions, it is crucial that the judgments that are implicit in such weighting be made as clear and comprehensible as possible, and thus be open to public scrutiny" (p. 6).

The human development index is a handy tool without any doubt but as Sen, one of the originators of this index, emphasizes the choice of weights is a sensible issue and ultimately a matter for social choice based on valuational arguments (Sen, 2003, p. 7). Sen goes one step further and stresses the time dimension. "When the ingredients of a judgment are diverse, any aggregate index with *constant* weights (the emphasis is by the author) over its diverse constituent elements would tend to oversimplify the evaluative exercise" (p. 12). One has to be interested in the present situation of countries but sometimes, changes over time are of particular interest. The spread of diseases as well as a more restricted access to clean water resources are important for life expectancy in developing countries. So a higher weight for these aspects would signal particular attention. In more developed countries where death at an early age is no longer a pressing issue, social exclusion measured by long–term unemployment may justify a higher weight in future investigations. Therefore, departures from the current structure and usage of the various indices may seem legitimate.

In this paper, we propose a particular way of measuring the standard of living available to an agent as well as to a whole country. The agent or country will be characterized by a capability set consisting of various vectors of functionings possible at any given time. The basis for our theoretical analysis is Lancaster's (1966) characteristics approach to consumer theory. In this approach consumer goods generate characteristics, and this is done according to a linear "input–output" relationship. The higher the income of a consumer or country, the higher are the maximally possible purchases of a particular good. However, in general, the consumer can choose among different consumer goods and, moreover, the consumer can spend part of his income on commodity a, let's say, another part on good b, a third part on commodity c, etc. In other words, combinations of different commodities are possible and income–wise feasible. In the space of characteristics, we obtain, due to the linear "production technology", star–shaped convex spaces.

In our context, we assume linear input–output relationships in a twofold way. Consumer goods (but also investment goods, like capital investments in land irrigation or education) generate characteristics and these characteristics lead to different functionings or functioning vectors. These represent health, longevity, literacy and other basic qualities. Given a particular income (for an individual) or a particular budget (for a country), the individual (or country, respectively) can acquire various consumer goods (a country would, additionally, run different investment projects). These yield various functioning vectors and combinations of these generate convex spaces of functionings. These spaces span the agent's as well as a country's capability set. Due to the underlying linearity, they are star–shaped.

The human development index (HDI) produces one real number for each country under investigation. By doing so, a complete ordering over all countries concerned is generated. Both the ordering as well as measured differences in the HDI, for example, between two countries a and b reveal deficiencies. Among the countries with high human development, an HDI value in 1999 of 0.939 for Norway and a value of 0.831 for Slovakia show quite a large gap between the two countries, whereas Slovakia and Hungary seem to be at a very similar stage of human development, the latter's HDI index being 0.829 for the same year.

In this paper, we do not consider indices or real numbers as indicators or benchmarks for comparisons. As has become clear above, we shall focus on *vectors of functionings*. In order to be judged living a satisfactory life, an agent or a country must have a given functioning vector in her capability set. We readily admit that determining such a reference functioning vector is, conceptually speaking, not easy. For the moment, we wish to assume that this problem has been solved (we just refer to the development and refinement of the HDI and other indices over the last twelve years). To improve her standard of living in terms of functionings, given the uncertainty associated with the development of society (and the world economy), it is not immediately clear along which direction the agent's or the country's functioning vector will grow as time progresses. Furthermore – and now we come back to Sen's remarks on constant weights and the aspect of changes over time, the reference functioning vector may, and perhaps should, change over time in its composition, paying, perhaps, more attention to the access to clean water resources and adult illiteracy in developing countries, and, perhaps, paying more attention to long–term unemployment and youth unemployment in more developed countries. We investigate how the agent's or country's standard of living may be measured, given these uncertainties within and among societies. We shall also examine the case where the reference functioning vector lies outside the capability set of the agent or the country considered.

The structure of the paper is as follows. Section 2 introduces the basic notation and definitions. Section 3 presents the axioms that we need for our first characterization result. Section 4 states this theorem and provides a proof. Section 5 introduces a deprivation–gap ordering and discusses a second result. Section 6 generalizes our previous approach by introducing a cone with a reference surface. The final section 7 is devoted to some concluding remarks.

2 Basic Notation and Definitions

Let $\mathbb{R}^n_+$ be the non-negative orthant of the n-dimensional real space. The vectors in $\mathbb{R}^n_+$ will be denoted by $x, y, z, a, b, \cdots$, and are interpreted as functioning vectors (Sen (1985, 1987)). For all $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in \mathbb{R}^n_+$, define $x \geq y$ as $x_i \geq y_i$ for all $i = 1, \ldots, n$, $x > y$ when $x_i \geq y_i$ for all $i = 1, \cdots, n$ and $x_j > y_j$ for some $j \in \{1, \cdots, n\}$, and $x >> y$ when $x_i > y_i$ for all $i = 1, \cdots, n$. For all $x, y \in \mathbb{R}^n_+$, we define the distance between them as follows: $||x - y|| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$.

At any given point of time, the set of all vectors that may be available to the individual is a subset of $\mathbb{R}^n_+$. Such a set will be called the individual's *capability set*. We will use A, B, C, etc. to denote the capability sets.

Our concern in this paper is to rank different capability sets in terms of the *standard of living* that they offer to the individual. In particular, we confine our attention to opportunity sets that are

(2.1) *compact*: a capability set $A \subseteq \mathbb{R}^n_+$ is compact iff A is closed and bounded,

(2.2) *convex*: a capability set $A \subseteq \mathbb{R}^n_+$ is convex iff, for all $x, y \in \mathbb{R}^n_+$ and all $\alpha \in [0, 1]$, if $x, y \in A$, then $\alpha x + (1 - \alpha)y \in A$,

(2.3) *star-shaped*: a capability set $A \subseteq \mathbb{R}^n_+$ is star-shaped iff, for all $x \in \mathbb{R}^n_+$ and all $t \in [0, 1]$, if $x \in A$, then $tx \in A$.

Let $\mathcal{K}$ be the set of all capability sets that are compact, convex and star-shaped. For all $A, B \in \mathcal{K}$, we write $A \subseteq B$ for "A being a subset of B" and $A \subset B$ for "A being a proper subset of B".

For all $A, B \in \mathcal{K}$ and all $x^* \in \mathbb{R}^n_+$, let $A >_{x^*} B$ denote: [whenever $x^* \in B$, there is a neighborhood, $\mathcal{N}(x^*, \epsilon) = \{x \in \mathbb{R}^n_+ : x \geq x^*, ||x - x^*|| \leq \epsilon\}$ where $\epsilon > 0$, of x^* such that $\mathcal{N}(x^*, \epsilon) \subseteq A$] and [for all $b \in B$ with $b > x^*$, there exists $a \in A$ such that $a >> b$]. Let $x^0 \in \mathbb{R}^n_{++}$ be the deprivation vector of functionings below which the individual's standard of living is judged to be "poor". Throughout sections 2–5, we assume that x^0 is fixed. For all $t > 0$, let

$$X(x^0, t) = \{x \in \mathbb{R}^n_+ : x \geq x^0, ||x - x^0|| \leq t\}.$$

Scalar t measures the Euclidean distance between two vectors in the functioning space. This, of course, presupposes that we can quantify each of the functionings appropriately so that there is a measurement scale common to all functionings considered. An analogy, but only an analogy, is a world where the utility functions of all individuals are fully cardinally comparable. Transformations would have to be positive affine with the same scale factor and the same change of origin for each functioning. Of course, other measures of distance could be considered as well, but we shall not do this in this paper.

For all $A \in \mathcal{K}$, let

$$r(A) = \begin{cases} -1 & \text{if } x^0 \notin A \\ \max_t\{t \in \mathbb{R}_+ : \{x \in \mathbb{R}^n_+ : x \geq x^0, ||x - x^0|| \leq t\} \subseteq A\} & \text{if } x^0 \in A \end{cases}$$

Figure 1 depicts the maximal $t \in \mathbb{R}_+$ for two capability sets A and B when $x^0 \in A \cap B$.

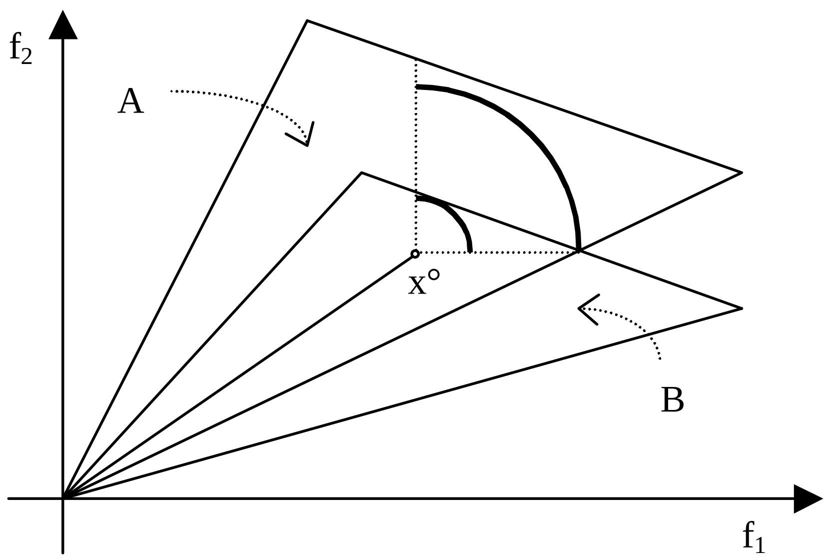

Figure 1: comparison of two capability sets A and B

Let $\succeq$ be a binary relation over $\mathcal{K}$ that satisfies *reflexivity:* [for all $A \in \mathcal{K}, A \succeq A$], *transitivity:* [for all $A, B, C \in \mathcal{K}$, if $A \succeq B$ and $B \succeq C$ then $A \succeq C$], and *completeness:* [for all $A, B \in \mathcal{K}$ with $A \neq B, A \succeq B$ or $B \succeq A$]. Thus, $\succeq$ is an *ordering*. The intended interpretation of $\succeq$ is the following: for all $A, B \in \mathcal{K}, [A \succeq B]$ will be interpreted as "*the degree of the standard of living offered by A is at least as great as the degree of the standard of living offered by B*". $\succ$ and $\sim$, respectively, are the asymmetric and symmetric part of $\succeq$.

3 Axiomatic Properties

In the following two sections, we present an axiomatic characterization of the standard of living ranking defined below:

$$\text{For all } A, B \in \mathcal{K}, A \succeq^r B \Leftrightarrow r(A) \geq r(B).$$

We begin by listing a set of axioms.

Definition 3.1. $\succeq$ over $\mathcal{K}$ satisfies

(3.1.1) **Monotonicity** iff, for all $A, B \in \mathcal{K}$, if $B \subseteq A$ then $A \succeq B$.

(3.1.2) **Betweenness** iff, for all $A, B \in \mathcal{K}$, if $A \succ B$ with $x^0 \in A \cap B$, then there exists $C \in \mathcal{K}$ such that $C >_{x^0} B$ and $A \succ C \succ B$.

(3.1.3) **Dominance** iff, for all $A, B \in \mathcal{K}$, if $x^0 \notin B$, then $A \succeq B$, and furthermore, if $x^0 \in A$, then $A \succ B$.

(3.1.4) **Domination in Terms of Uncertain Development** iff, for all $A, B \in \mathcal{K}$, if there exists $t > 0$ such that $X(x^0, t) \cap A = X(x^0, t)$, and $B \cap X(x^0, t) \subset X(x^0, t)$, then $A \succ B$.

The intuition behind Monotonicity is simple and easy to explain. It requires that whenever B is a subset of A, then A is ranked at least as high as B concerning standards of living offered. Betweenness requires that when A is judged to offer a higher standard of living than B relative to the deprivation vector x^0, there must exist a set C such that $C >_{x^0} B$ and A offers a higher standard of living than C, which in turn offers a higher standard of living than B. Dominance requires that whenever the deprivation vector x^0 is not achievable in B, the standard of living offered by B cannot be higher than that offered by any other capability set A, and furthermore, if the deprivation vector x^0 is achievable under A, then A offers a higher standard of living than B. Domination in Terms of Uncertain Development requires that, for two capability sets A and B, whenever A results from progress made in all dimensions of functioning vectors, while B does not offer this particular kind of progress, the standard of living under A is judged to be higher than that offered by B.

4 A First Characterization Result

Theorem 4.1. Suppose $\succeq$ over $\mathcal{K}$ is an ordering. Then, $\succeq$ satisfies Monotonicity, Betweenness, Dominance, and Domination in Terms of Uncertain Development if and only if $\succeq = \succeq^r$.

Proof. It can be checked that $\succeq^r$ is an ordering and satisfies Monotonicity, Betweenness, Dominance and Domination in Terms of Uncertain Development.

We now show that if $\succeq$ over $\mathcal{K}$ satisfies Monotonicity, Betweenness, Dominance and Domination in Terms of Uncertain Development, then $\succeq = \succeq^r$.

(i) We first show that, for all $t > 0$ and all $A, B \in \mathcal{K}$, if $r(A) = t = r(B)$ and $B \cap X(x^0, t) = X(x^0, t) = A \cap X(x^0, t)$, then $A \sim B$. Suppose $A \succ B$. Then, by Betweenness, there exists $C \in \mathcal{K}$ such that $C >_{x^0} B$, and $A \succ C \succ B$. Since $r(B) = t > 0$, $C >_{x^0} B$, B and C are compact and star-shaped, and for some positive $t' > t$ and some set $C' \in \mathcal{K}$, $\{x \in \mathbb{R}^n_+ : x \geq x^0, x \in C'\} = X(x^0, t')$, $C' \subseteq C$, and $B \cap X(x^0, t') \subset X(x^0, t')$. By Monotonicity, $C \succeq C'$ and by Domination in Terms of Uncertain Development (henceforth, Domination for short), $C' \succ B$. Hence, $A \succ C' \succ B$. Noting that $r(A) = t < t' = r(C')$, we must have $X(x^0, t) \cap C'$ is a proper subset of $C' \cap X(x^0, t') = X(x^0, t')$. By Domination, since $A \cap X(x^0, t') \subset X(x^0, t')$, we obtain $C' \succ A$ which is in contradiction to $A \succ C'$. Therefore, it is not true that $A \succ B$. Similarly, it can be shown that it is not true $B \succ A$. Therefore, $A \sim B$.

(ii) Second, we show that for all $A, B \in \mathcal{K}$, if $r(A) > r(B) > 0$, then $A \succ B$. Let $A, B \in \mathcal{K}$ be such that $r(A) > r(B) > 0$. Consider $A', B' \in \mathcal{K}$ such that $A' \cap X(x^0, r(A)) = X(x^0, r(A)), B' \cap X(x^0, r(B)) = X(x^0, r(B))$ and $B' \subset A'$. Since $r(A) > r(B) > 0$, such A' and B' exist. From (i), $A' \sim A$ and $B' \sim B$. By Domination, $A' \succ B'$. Then, $A \succ B$ follows from transitivity of $\succeq$.

(iii) Third, we show that for all $A, B \in \mathcal{K}$, if $r(A) > 0 = r(B)$, then $A \succ B$. Note that, since $r(A) > 0 = r(B)$, it must be true that $B \cap X(x^0, r(A)) \subset A \cap X(x^0, r(A)) = X(x^0, r(A))$. By Domination, $A \succ B$ follows easily.

(iv) We next show that, for all $A, B \in \mathcal{K}$, if $x^0 \notin A$ and $x^0 \notin B$, then $A \sim B$. Since $x^0 \notin A$, by Dominance, $B \succeq A$. Similarly, by Dominance, from $x^0 \notin B$, it follows that $A \succeq B$. Therefore, $A \sim B$.

(v) We now show that for all $A, B \in \mathcal{K}$, if $B = \{x \in \mathbb{R}^n_+ : x = tx^0, \forall t \in [0, 1]\}$, $x^0 \in A$ and $r(A) = 0$, then $A \succ B$ for $t < 1$ and $A \sim B$ for $t = 1$. For the first case, since $x^0 \in A$ and $x^0 \notin B$, we obtain, by Dominance, that $A \succ B$. Consider now the case that $t = 1$. Then $x^0 \in B$ and $x^0 \in A \cap B$. First, since $A \in \mathcal{K}$, clearly, $B \subseteq A$. By Monotonicity, $A \succsim B$. Suppose that $A \succ B$. Then, by Betweenness, there exists $C \in \mathcal{K}$ such that $C >_{x^0} B$ and $A \succ C \succ B$. Note that, since $C >_{x^0} B$ and $x^0 \in B$, $r(C) > 0$. From (iii) above and $r(A) = 0$, $C \succ A$ follows immediately, which is a contradiction to $A \succ C$ obtained earlier. Therefore, $A \sim B$.

(vi) From (v), for all $A, B \in \mathcal{K}$, if $x^0 \in A \cap B$ and $r(A) = r(B) = 0$, then $A \sim B$ follows immediately.

(vii) To complete the proof, we note that, for all $A, B \in \mathcal{K}$, if $x^0 \in A$ and $x^0 \notin B$ and $r(A) = 0$, by Dominance, $A \succ B$.

Therefore, (i) – (vii), together with the transitivity of $\succeq$ complete the proof of Theorem 4.1. ■

5 Further Properties and A Deprivation-Gap Ordering

The ordering $\succeq^r$ defined in Section 3 and characterized in Section 4 ranks all capability sets $A, B \in \mathcal{K}$ with $x^0 \notin (A \cup B)$ equally in terms of the standard of living. This is rather unsatisfactory. In this section, we first propose a ranking rule that avoids this undesirable feature. We then propose several properties to characterize this new ranking rule.

For all $A, B \in \mathcal{K}$, $A > B$ is to denote that, [for all $b \in B$ with $b >> 0$, there exists a neighborhood $\mathcal{N}(b, \epsilon) = \{x \in \mathbb{R}^n_+ : x \geq b, ||x - b|| \leq \epsilon\}$ where $\epsilon > 0$ of b such that $\mathcal{N}(b, \epsilon) \subseteq A$] and [for all $b \in B$, there exists $a \in A$ such that $a >> b$]. Note that whenever $A > B$, then $A >_{x^0} B$.

For all $t > 0$ and all $x^0 \in \mathbb{R}^n_+$, let $O(x^0, t) = \{x \in \mathbb{R}^n_+ : ||x - x^0|| \leq t\}$.

For all $A \in \mathcal{K}$, let

$$r^*(A) = \begin{cases} -\min_t\{t \in \mathbb{R}_+ : \{x \in \mathbb{R}^n_+ : ||x - x^0|| \leq t\} \cap A \neq \emptyset\} & \text{if } x^0 \notin A \\ \max_t\{t \in \mathbb{R}_+ : \{x \in \mathbb{R}^n_+ : x \geq x^0, ||x - x^0|| \leq t\} \subseteq A\} & \text{if } x^0 \in A \end{cases}$$

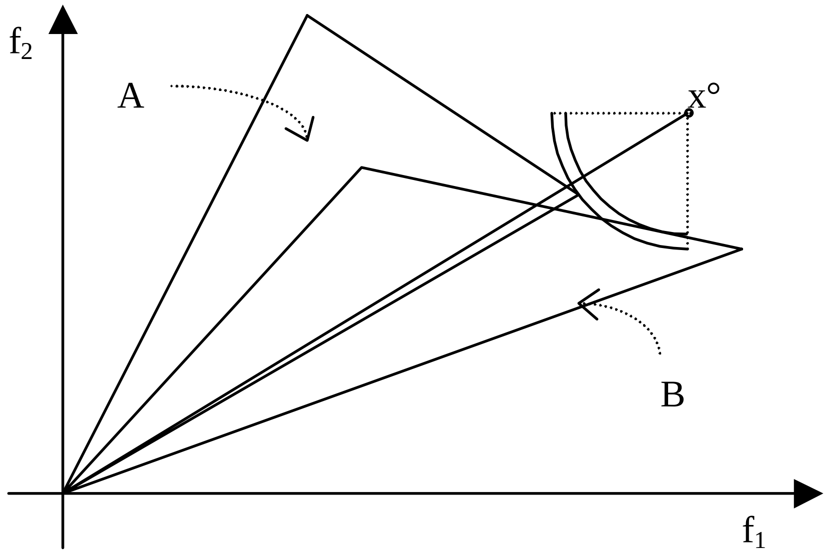

Figure 2: comparison of two capability sets A and B in terms of a deprivation–gap ordering

Figure 2 depicts the minimal $t \in \mathbb{R}_+$ for two capability sets A and B when $x^0 \notin A \cup B$.

Define the following *deprivation-gap ordering*:

$$\text{For all } A, B \in \mathcal{K}, A \succeq^{r^*} B \Leftrightarrow r^*(A) \geq r^*(B).$$

Consider the following axioms:

Definition 5.1. $\succeq$ over $\mathcal{K}$ satisfies

(5.1.1) **Strong Betweenness** iff, for all $A, B \in \mathcal{K}$, if $A \succ B$, then there exists $C \in \mathcal{K}$ such that $C > B$ and $A \succ C \succ B$.

(5.1.2) **Regressive Domination** iff, for all $A, B \in \mathcal{K}$, if there exists $t > 0$ such that $O(x^0, t) \cap A \neq \emptyset$, and $B \cap O(x^0, t) = \emptyset$, then $A \succ B$.

Strong Betweenness requires that if the standard of living offered by a capability set A is higher than the standard of living offered by a capability set B, then there always exists a capability set C which has $C > B$ and which offers a standard of living between A and B. Given that $A > B$ implies that $A >_{x^0} B$, it is straightforward to check that Strong Betweenness implies Betweenness. Regressive Domination is the counterpart of Domination in terms of Uncertain Development, and deals with the situation in which there is a possibility of "regressive development": if the capability set A dominates the capability set B in the fashion of "regressive development", then A offers a higher standard of living than B. It can be checked that Regressive Domination implies that, whenever $x^0 \in A$ while $x^0 \notin B$, we must have $A \succ B$. Thus, Regressive Domination is a stronger requirement than Dominance proposed in Section 3.

Theorem 5.2. Suppose $\succeq$ over $\mathcal{K}$ is an ordering. Then, $\succeq$ satisfies Monotonicity, Strong Betweenness, Domination in Terms of Uncertain Development and Regressive Domination if and only if $\succeq = \succeq^{r^*}$.

Proof. We first note that $\succeq^{r^*}$ satisfies Monotonicity, Strong Betweenness, Domination in terms of Uncertain Development and Regressive Domination. Therefore, we need to show if $\succeq$ satisfies Monotonicity, Strong Betweenness, Domination in Terms of Uncertain Development and Regressive Domination, then $\succeq = \succeq^{r^*}$.

Let $\succeq$ be an ordering that satisfies the four properties specified in Theorem 5.2. Note that, since Strong Betweenness implies Betweenness, and Regressive Domination implies Dominance, from the proof of Theorem 4.1, the following must be true:

(*) for all $A, B \in \mathcal{K}$, if $x^0 \in A$ and $x^0 \notin B$, then $A \succ B$, and
if $x^0 \in A \cap B$, then $r^*(A) \geq r^*(B) \Leftrightarrow A \succeq B$.

Therefore, it remains to be shown that, if $x^0 \notin A \cup B$, then $r^*(A) \geq r^*(B) \Leftrightarrow A \succeq B$.

Let $A, B \in \mathcal{K}$ be such that $x^0 \notin A \cup B$ and $r^*(A) = r^*(B)$. Clearly, $r^*(A) < 0$ since A is closed, compact, star-shaped, and $x^0 \notin A$. For such A and B, we need to show that $A \sim B$. Suppose to the contrary that $A \succ B$ or $B \succ A$. If $A \succ B$, by Strong Betweenness, there exists $C \in \mathcal{K}$ such that $C > B$ and $A \succ C \succ B$. Note that, since $C > B$, there exists a positive number $t < -r^*(A)$ such that $O(x^0, t) \cap C \neq \emptyset$. Since $-r^*(A) > t$, it must be the case that $A \cap O(x^0, t) = \emptyset$. By Regressive Domination, $C \succ A$, a contradiction. Similarly, $B \succ A$ leads to a similar contradiction. Therefore, $A \sim B$.

Next, for all $A, B \in \mathcal{K}$, if A, B are such that $x^0 \notin A \cup B$, $r^*(A) > r^*(B)$, then $A \succ B$ follows directly from Regressive Domination.

Therefore, for all $A, B \in \mathcal{K}$, if $x^0 \notin A \cup B$, then $A \succeq B \Leftrightarrow r^*(A) \geq r^*(B)$. This, together with (*), proves Theorem 5.2. ■

6 Generalization: The ϵ–Cone

Up to this point, it was assumed that the reference vector of functionings is a point on a straight line from the origin of the functioning space. This concept is easy to define and relatively easy to "handle". But is it realistic? Should the "point" of reference perhaps be a set of points so that different combinations of functionings would be equally suitable as points of reference? Once this idea is introduced, it would be possible to consider "trade–off" relationships between functionings so that we would have various combinations of functionings available which would be equivalent in terms of serving as appropriate reference points. To be somewhat more explicit, let us take health, education and being nourished as basic functionings. A better education will teach people to avoid certain illnesses, also a higher quality of food intakes will reduce health outlays so that in terms of functionings there seem to be "efficiency–frontiers" in the functioning space.

And was it justified to treat the path to reference vector x^0 as a straight line from the origin or would it be more appealing to conceive of this path as "somewhat meandering" in the north–east direction?

Both points can be taken care of by introducing, what we shall call, an ϵ–cone stretching north–east with its vertex at the origin. A cone with angle ϵ and length λ has the property that its north–east boundary with all points having length λ from the origin constitutes a (reference) surface. This surface with concave curvature looks like an efficiency frontier. Its north–east boundary will extend in length, the larger λ which means that trade–offs among functioning vectors are possible over a wider range. This seems plausible since countries with higher levels of functionings apparently have more trade–off possibilities than poorer countries (with lower levels of functionings). Geometrically, λ can also be interpreted as the radius of a virtual circle from the origin, where only a small segment in direction north–east is considered. The larger the radius, the greater the absolute values of the components of the functioning vectors can get that belong to the ϵ–cone. Turned around and viewed from the aspect of deprivation level, the larger λ, the higher are the demands for minimally required vectors of functionings. The latter is a feature that was, of course, already true in the case of a straight line. Note that when the angle of the cone converges to zero, we are back to our original approach. The cone itself can be interpreted as the envelope of the path of reference points over time.

We want to become somewhat more formal now.

Let $\epsilon > 0$ and $\lambda > 0$. Consider a cone with angle ϵ, vertex 0 and length λ. We shall call such a cone an ϵ-cone. From now on, an ϵ-cone with length of λ will be denoted by $\epsilon(\lambda)$. In the following discussion, $\epsilon(\lambda)$ is taken as given. Let $\partial\epsilon(\lambda) := \{x \in \epsilon(\lambda) : \text{there exists no } y \in \epsilon(\lambda) \text{ such that } y > x\}$. Let $x_i \in \partial\epsilon(\lambda)$. For any $t \geq 0$, define $X(\epsilon(\lambda), x_i, t) := \{x \in \mathbb{R}^n_+ : x \geq x_i, ||x - x_i|| \leq t\}$.

For any capability set A and any $x_i \in \partial\epsilon(\lambda)$, let $t(A, x_i) = \max\{t : X(\epsilon(\lambda), x_i, t) \subseteq A\}$. For any capability set A and any $x_i \in \partial\epsilon(\lambda)$, let

$A(x_i, t(A, x_i)) = X(\epsilon(\lambda), x_i, t(A, x_i))$.

For any $\epsilon(\lambda)$ and any $j = 1, \ldots, n$, let $\epsilon_j^*(\lambda) = \min\{x_i^j | x_i \in \mathbb{R}_+^n : x_i \in \partial\epsilon(\lambda)\}$ and let $\epsilon^*(\lambda) = (\epsilon_1^*(\lambda), \ldots, \epsilon_j^*(\lambda), \ldots, \epsilon_n^*(\lambda))$. For any $t \geq 0$ and $\epsilon(\lambda)$, we define $X(\epsilon^*(\lambda), t) = \{x \in \mathbb{R}_+^n : ||x - \epsilon^*(\lambda)|| \leq t,\ x \geq \epsilon^*(\lambda)\}$.

In this section, we propose two new methods of ranking capability sets, based on the ϵ–cone. Consider the first one:

(1) For any capability set A and $\epsilon(\lambda) \subseteq A$, let $\hat{s}(A) = \max\{t \in \mathbb{R}_+ : X(\epsilon^*(\lambda), t) \subseteq A\}$. Define $\hat{r}(A) = \hat{s}(A)$ if $\hat{s}(A) \geq 0$ and $\hat{r}(A) = -1$ if otherwise. Then, for all capability sets A and B,

$$A \succeq_{\epsilon^*} B \leftrightarrow \hat{r}(A) \geq \hat{r}(B) .$$

Let us explain in more detail what we have done in the last two paragraphs. Given all $x_i \in \partial\epsilon(\lambda)$, we have constructed a point that consists of minimal components in all n dimensions. Due to the construction of the ϵ–cone, the point lies inside the cone. This point is then taken as the point of reference for finding the largest circle such that its segment in north–east direction still lies in capability set A. In many respects we are now back to sections 2–4, since our reference vector has again become a single point. Instead of looking for the *minimal* coordinates of all points in $\partial\epsilon(\lambda)$, we could have also determined the *maximal* coordinates for all dimensions of functionings. This point would lie outside the ϵ–cone (see Figure 3 below).

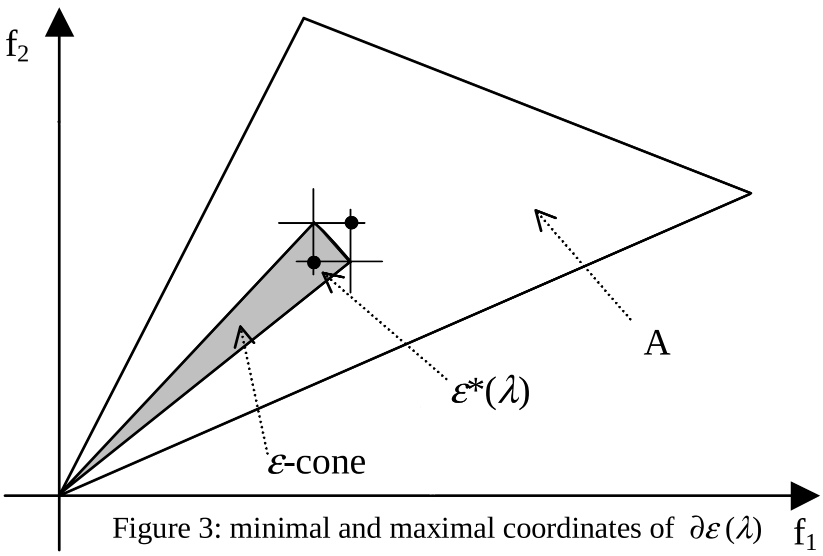

Figure 3: minimal and maximal coordinates of $\partial\varepsilon(\lambda)$

We just asserted that for this particular proposal we are, to a great extent, back to our first approach. A characterization of this proposal would largely follow our first characterization result in section 4. Therefore, we omit further details.

(2) The second proposal concerns a dominance relationship. The situation is depicted in Figure 4.

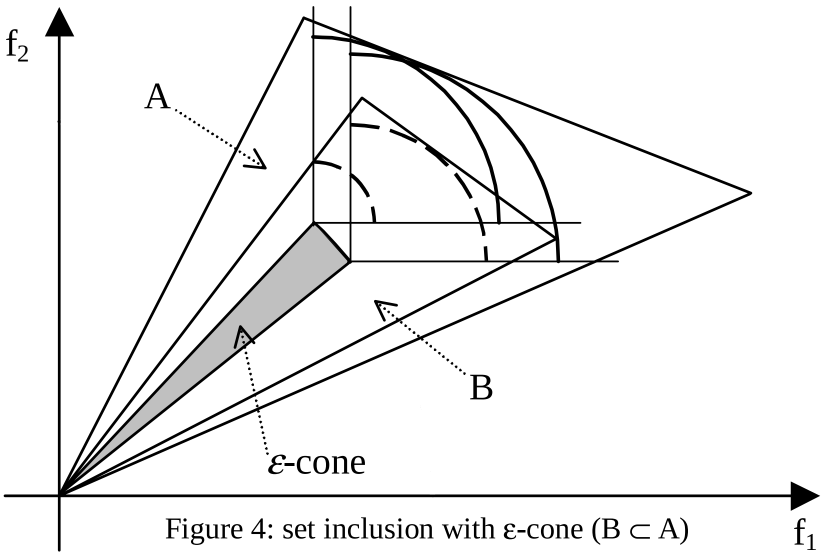

Figure 4: set inclusion with ε-cone (B ⊂ A)

First, we introduce set $X(x_i, t)$. For all $t \geq 0$ and $x_i \in \partial\epsilon(\lambda)$, let

$$X(x_i, t) = \{x \in \mathbb{R}^n_+ : x \geq x_i, ||x - x_i|| \leq t\}.$$

For any $A, B \in \mathcal{K}$, we say that B is a proper subset of A relative to $\epsilon(\lambda)$, to be denoted by $B \subset_{\epsilon(\lambda)} A$, if B is a subset of A and for all $t \geq 0$ and all $x_i \in \partial\epsilon(\lambda)$, $X(x_i, t) \subseteq B \rightarrow X(x_i, t) \subseteq A$, and for some $t' \geq 0$ and some $x_i' \in \partial\epsilon(\lambda)$, $X(x_i', t') \subseteq A$ and $X(x_i', t')$ is not a subset of B.

Define $\succeq_{dominance}$ over $\mathcal{K}$ as follows: for all $A, B \in \mathcal{K}$,

if $\partial\epsilon(\lambda) \cap B = \partial\epsilon(\lambda) \cap A = \emptyset$, then $A \sim_{dominance} B$,

if $\partial\epsilon(\lambda) \cap B = \emptyset, \partial\epsilon(\lambda) \cap A \neq \emptyset$, then $A \succ_{dominance} B$,

if $\partial\epsilon(\lambda) \cap B \neq \emptyset, \partial\epsilon(\lambda) \cap A \neq \emptyset$, then $A \succeq_{dominance} B \Leftrightarrow [B(x_i, t(B, x_i)) \subseteq A(x_i, t(A, x_i))$ for all $x_i \in \partial\epsilon(\lambda)]$.

It is clear from this definition that the dominance relationship can only yield a partial ordering.

Definition 6.1. $\succeq$ over $\mathcal{K}$ satisfies

Cone Dominance iff, for all $A, B \in \mathcal{K}$,

if $\partial\epsilon(\lambda) \cap B = \emptyset$, then $A \succeq B$,

if $\partial\epsilon(\lambda) \cap B = \emptyset$ and $\partial\epsilon(\lambda) \cap A \neq \emptyset$, then $A \succ B$.

Strong Domination in Terms of Uncertain Development iff, for all $A, B \in \mathcal{K}$,

if for all $t \geq 0$ and all $x_i \in \partial\epsilon(\lambda)$, $X(x_i, t) \subset B \rightarrow X(x_i, t) \subseteq A$, then $A \succeq B$,

if for some $t \geq 0$ and some $x_i \in \partial\epsilon(\lambda)$, $X(x_i, t)$ is a subset of A but $X(x_i, t)$ is not a subset of B, then $A \succeq B \rightarrow A \succ B$.

Strict Betweenness iff, for all $A, B \in \mathcal{K}$, if $A \succ B$, then there exists $C \in \mathcal{K}$ such that $C \subset_{\epsilon(\lambda)} A$, $B \subset_{\epsilon(\lambda)} C$ and $A \succ C \succ B$.

Theorem 6.1. Suppose that $\succeq$ is reflexive and transitive. Then, $\succeq$ satisfies Cone Dominance, Strong Domination in Terms of Uncertain Development, and Strict Betweenness if and only if $\succeq = \succeq_{dominance}$.
Proof. It can be checked that $\succeq_{dominance}$ is reflexive and transitive and satisfies Cone Dominance, Strong Domination in Terms of Uncertain Development, and Strict Betweenness. We now show that if $\succeq$ over $\mathcal{K}$ satisfies Cone Dominance, Strong Domination in Terms of Uncertain Development, and Strict Betweenness, then $\succeq = \succeq_{dominance}$.

Let $A, B \in \mathcal{K}$. We consider the following cases.

(i) $\partial\epsilon(\lambda) \cap A = \partial\epsilon(\lambda) \cap B = \emptyset$. Since $\partial\epsilon(\lambda) \cap A = \emptyset$, by Cone Dominance, $B \succeq A$. Similarly, since $\partial\epsilon(\lambda) \cap B = \emptyset$, by Cone Dominance, $A \succeq B$. Therefore, $A \sim B$ follows immediately.

(ii) $\partial\epsilon(\lambda) \cap A \neq \emptyset$ and $\partial\epsilon(\lambda) \cap B = \emptyset$. By Cone Dominance, from $\partial\epsilon(\lambda) \cap A \neq \emptyset$ and $\partial\epsilon(\lambda) \cap B = \emptyset$, it follows that $A \succ B$.

(iii) $\partial\epsilon(\lambda) \cap A = \emptyset$ and $\partial\epsilon(\lambda) \cap B \neq \emptyset$. In this case, $B \succ A$ follows from a similar argument as in (ii).

(iv) $\partial\epsilon(\lambda) \cap A \neq \emptyset$ and $\partial\epsilon(\lambda) \cap B \neq \emptyset$.

Consider first that $[B(x_i, t(B, x_i)) \subseteq A(x_i, t(A, x_i))$ for all $x_i \in \partial\epsilon(\lambda)]$ and $[A(x_i, t(A, x_i)) \subseteq B(x_i, t(B, x_i))$ for all $x_i \in \partial\epsilon(\lambda)]$. By Strong Domination in Terms of Uncertain Development, we obtain $A \succeq B$ and $B \succeq A$, that is, $A \sim B$.

Next, consider $[B(x_i, t(B, x_i)) \subseteq A(x_i, t(A, x_i))$ for all $x_i \in \partial\epsilon(\lambda)]$ and [for some $t \geq 0$ and $x_0 \in \partial\epsilon(\lambda)$, $X(x_0, t)$ is a subset of A but $X(x_0, t)$ is not a subset of B]. Note that, in this case, by the first part of Strong Domination in Terms of Uncertain Development,

$A \succeq B$. Then, by the second part of Strong Domination in Terms of Uncertain Development, $A \succ B$ follows easily.

Similarly, when $[A(x_i, t(A, x_i)) \subseteq B(x_i, t(B, x_i))$ for all $x_i \in \partial\epsilon(\lambda)]$ and [for some $t \geq 0$ and $x_0 \in \partial\epsilon(\lambda)$, $X(x_0, t)$ is a subset of B but $X(x_0, t)$ is not a subset of A], $B \succ A$ follows immediately.

Finally, consider $[B(x_1, t(B, x_1))$ is a proper subset of $A(x_1, t(A, x_1))$ for some $x_1 \in \partial\epsilon(\lambda)]$, and, $[A(x_2, t(A, x_2))$ is a proper subset of $B(x_2, t(B, x_2))$ for some $x_2 \in \partial\epsilon(\lambda)]$. We need to show that not$(A \succeq B)$ and not$(B \succeq A)$.

If $A \sim B$, by Strong Domination in Terms of Uncertain Development, from $[B(x_1, t(B, x_1))$ is a proper subset of $A(x_1, t(A, x_1))$ for some $x_1 \in \partial\epsilon(\lambda)]$ implying $X(x_1, t(A, x_1))$ is a subset of A and $X(x_1, t(A, x_1))$ is not a subset of B, we must have $A \succ B$, a contradiction.

From $A \sim B$, by Strong Domination we arrive at $B \succ A$ via an analogous argument, again a contradiction.

If $A \succ B$, by Strict Betweenness, there exists $C \in \mathcal{K}$ such that $B \subset_{\epsilon(\lambda)} C$, $C \subset_{\epsilon(\lambda)} A$, and $A \succ C \succ B$. Note that $[B(x_1, t(B, x_1))$ is a proper subset of $A(x_1, t(A, x_1))$ for some $x_1 \in \partial\epsilon(\lambda)]$, and, $[A(x_2, t(A, x_2))$ is a proper subset of $B(x_2, t(B, x_2))$ for some $x_2 \in \partial\epsilon(\lambda)]$. Therefore, A is not a subset of B and B is not a subset of A in this case. Note, however, that C is proper subset of A and B is a proper subset of C. Hence, B must be a proper subset of A, a contradiction.

Following a similar argument, $B \succ A$ leads to another contradiction.

Therefore, it must be the case that not$(A \succeq B)$ and not$(B \succeq A)$.

This completes the proof of Theorem 6.1. ■

7 Concluding Remarks

In this paper we have proposed several new ways to measure the standard of living and to compare the standards of living of two persons or, more importantly, two countries. The basis for our approach is Sen's proposal to consider functioning vectors and capability sets. The human development index constructs a real number for each country under investigation. Comparisons among different countries are done by calculating numerical differences of their respective development index. The issue of determining the appropriate weights for those components that enter a development index is central for constructing this index. These weights can and will change over time. Consequently, the aggregate real number will vary under different weighting schemes. In this paper, comparisons among different countries are based on a reference functioning vector that may change as time progresses or on a reference surface. The different functionings that constitute the reference vector undergo a re–evaluation over time, also in relation to each other. Functionings are the focus of attention in many investigations on human development these days. Therefore, we think that the approach formulated here can be used for real–world applications. It should be interesting to see how the rankings according to the currently used HDI would fare in comparison with our new measures.

To conclude this paper, two remarks are in order. First, due to our assumption of the linear "production technology" in producing functioning vectors, we have focused on capability sets that are compact, convex and star-shaped. We realize that the linear "production technology" is a restrictive assumption. It would, therefore, be interesting to relax this assumption and examine the problem of ranking capability sets thus obtained in terms of standards of living offered. Secondly, it is implicitly assumed, given the uncertainty associated with the development of society, that directions along which the agent's or the country's functioning vector will grow as time progresses are "equally likely". In other words, we admitted the full 90–degree–angle starting from the reference point in north–east direction. One may argue that, even though it is not possible to know the precise direction along which the country's functioning vector will grow as time goes by, a range of possible directions can be identified and this range is much narrower than the range of all possible directions that we assumed in our framework. One could, for example, limit the width of the angle due to information from the past development of more advanced countries. This, of course, presupposes that countries at a lower stage of development will "approximately" follow the path of countries more advanced. It would then be interesting to explore ways of measuring the standard of living offered by capability sets when the information about the range of possible growth directions becomes available. But we leave this point for another occasion.

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