

On Hierarchical Dictators*

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Abstract

This paper explores implications of a social welfare function satisfying the Strong Pareto Principle (SPP) and Independence of Irrelevant Alternatives (IIA) under various domain restrictions. It is shown that, for certain domain restrictions, any social welfare function satisfying SPP and IIA must be hierarchically dictatorial: there exists a dictator; when the dictator is indifferent between two social states x and y , there exists another individual whose strict preference over x and y is preserved by the social preference; and when these two individuals are indifferent between x and y , there exists a third individual whose strict preference over x and y is preserved by the social preference; and so on.

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by Yongsheng Xu

In his classical book, Arrow (1951; 1963) shows that if a social welfare function satisfies the condition of Unrestricted Domain, the Weak Pareto principle and the condition of Independence of Irrelevant Alternatives, then it must be dictatorial: there exists a dictator d such that whenever d has a strict preference over a pair of social states x and y , the social preference over this pair of social states must be exactly the dictator d 's strict preference over x and y . Very little can be said about social preference over a pair a and b when the dictator d is indifferent between a and b . The purpose of this paper is two-fold. First, we offer a precise account on social preference over a pair a and b when the dictator d is indifferent between them. We show that, in an Arrovian framework, if we replace the weak Pareto principle by the strong Pareto principle and keep Unrestricted Domain and Independence of Irrelevant Alternatives (IIA) intact, any social welfare function must take the following form of *hierarchical dictatorship*. In an n -individual society, individuals can be arranged in such a way that person i is the level- i dictator: person 1 is the level-1 dictator who is the dictator in the sense of Arrow, person 2 is the level-2 dictator in the sense that whenever level-1 dictator is indifferent between two social states x and y and person 2 has a strict preference over x and y , then the social preference over x and y coincides with person 2's strict preference, and person i is the level- i dictator whose strict preference over x and y is the social preference over x and y whenever level-1 dictator, level-2 dictator, \dots , level- $(i - 1)$ dictator are indifferent between x and y .¹ Secondly, we discuss the relationship between the domain restrictions for obtaining hierarchical dictatorship of a social welfare function satisfying IIA and strong Pareto and the domain restriction that, in the presence of IIA and weak Pareto, gives rise to a dictatorial social welfare function. We point out that, in general, these two domain restrictions are not identical: the domain restriction on a social welfare function satisfying IIA and weak Pareto to yield a dictatorial social welfare function cannot guarantee a social welfare function satisfying IIA and strong Pareto to be hierarchically dictatorial. We also provide a simple condition on the domain of a

¹It should be noted that the result on hierarchical dictators when a social welfare function satisfies Unrestricted Domain, strong Pareto and IIA is not new. Indeed, a similar result on hierarchical dictators is discussed and proved in Craven (1992) (Theorem 3.10).

social welfare function f so that when f is dictatorial when IIA and weak Pareto are satisfied, it must also be hierarchically dictatorial in the presence of IIA and strong Pareto.

To set the stage for our discussion, we first introduce some notation and definitions. Let X be the finite set of all conventionally defined social states, which are mutually exclusive and jointly exhaustive. It is assumed that X contains more than 2 alternatives. The alternatives of X are denoted by x, y, z, \dots .

Let $N = \{1, 2, \dots, n\}$ be the finite set of all individuals in the society, where $2 \leq n = \#N$. Each individual $i \in N$ is assumed to have a preference ordering R_i over X , which is *reflexive*, *complete* and *transitive*. The asymmetric part and the symmetric part of R_i are denoted by $P(R_i)$ and $I(R_i)$, respectively, which denote the strict preference relation and the indifference relation of $i \in N$.

Let φ be the set of all logically possible orderings over X . Then a *profile* $\mathbf{R} = (R_1, R_2, \dots, R_n)$ of individual preference orderings, one ordering for each individual, is an element of φ^n . A *social welfare function* (SWF) is a function f which maps each and every profile in some subset D_f of φ^n into φ . When $R = f(\mathbf{R})$ holds for some $\mathbf{R} \in D_f$, $I(R)$ and $P(R)$ stand, respectively, for the social indifference relation and the social strict preference relation corresponding to R .

For all $i \in N$, we say that i is the level-1 dictator iff for all $x, y \in X$ and all $\mathbf{R} = (R_1, \dots, R_n) \in D_f$, if $xP(R_i)y$ then $xP(R)y$, where $R = f(\mathbf{R})$. Thus, the level-1 dictator is the dictator in the sense of Arrow. With the help of level-1 dictator, we can now define the following concept. For all $i \in N$ and for all $k \in \{2, \dots, n\}$, we say that i is the level- k dictator iff for all $x, y \in X$ and all $\mathbf{R} = (R_1, \dots, R_n) \in D_f$, if $[xI(R_j)y$ for $j = 1, \dots, k-1$ and $xP(R_k)y]$ then $xP(R)y$, where $R = f(\mathbf{R})$. Clearly, the level- k dictator dictates SWF over a pair of social states when the level-1 dictator, level-2 dictator, \dots , and level- $(k-1)$ are indifferent between this pair of social states. It should be noted that for $i \neq j$, the level- i dictator and the level- j dictator cannot be the same individual.

An SWF is said to be *hierarchically dictatorial* iff there exists a permutation σ of N such that for all $i \in N$, $\sigma(i)$ is the level- i dictator, and for all $x, y \in X$ and all $\mathbf{R} = (R_1, \dots, R_n) \in D_f$, $[xI(R_i)y$ for all $i \in N] \Rightarrow xI(R)y$ where $R = f(\mathbf{R})$.

Clearly, a hierarchically dictatorial SWF yields a reflexive, transitive and complete social preference.

We now introduce several conditions on f which will be used for our subsequent

discussions. They are well known in the literature and we omit discussions.

Weak Pareto Principle (WPP): For all $x, y \in X$, and for all $\mathbf{R} = (R_1, R_2, \dots, R_n) \in D_f$, if $xP(R_i)y$ holds for all $i \in N$, then we have $xP(R)y$, where $R = f(\mathbf{R})$.

Strong Pareto Principle (SPP): For all $x, y \in X$, and for all $\mathbf{R} = (R_1, R_2, \dots, R_n) \in D_f$, if xR_iy holds for all $i \in N$, then we have xRy ; and if xR_iy holds for all $i \in N$ and $xP(R_i)y$ holds for at least one $i \in N$, then we have $xP(R)y$, where $R = f(\mathbf{R})$.

Independence of Irrelevant Alternatives (IIA): For all $\mathbf{R}^1 = (R_1^1, R_2^1, \dots, R_n^1)$, $\mathbf{R}^2 = (R_1^2, R_2^2, \dots, R_n^2) \in D_f$, and for all $x, y \in X$, if $[xR_i^1y \Leftrightarrow xR_i^2y]$ holds for all $i \in N$, then $[xR^1y \Leftrightarrow xR^2y]$ holds, where $R^1 = f(\mathbf{R}^1)$ and $R^2 = f(\mathbf{R}^2)$.

Theorem 1. A social welfare function on $D_f = \wp^n$ is hierarchically dictatorial if and only if it satisfies SPP and IIA.

Proof. Though a similar result is proved in Craven (1992), we present an independent proof which is somewhat different from and more detailed than his proof. Let $D_f = \wp^n$. It can be checked that if a social welfare function on D_f is hierarchically dictatorial, then it satisfies SPP and IIA. We now show that if a social welfare function f on $D_f = \wp^n$ satisfies SPP and IIA, then it must be hierarchically dictatorial.

Let $N = \{1, 2, \dots, n\}$ and f be a social welfare function on D_f . Suppose f satisfies SPP and IIA. To highlight the subtle nature of the domain restriction for obtaining a hierarchically dictatorial social welfare function, we prove the following claim first:

Claim 2. Let N' be a proper subset of N . There exists $k \in N - N'$ such that, for all $x, y \in X$ and all $\mathbf{R} = (R_1, \dots, R_n) \in D_f$, if $xI(R_i)y$ for all $i \in N'$, then $xP(R_k)y \Rightarrow xP(R)y$, where $R = f(\mathbf{R})$.

Before we present the proof of Claim 2, we define a group of individuals $N_1 \subset N - N'$ being N' -conditionally decisive over a pair x and y , where $x, y \in X$, as follows: for all $\mathbf{R} = (R_1, \dots, R_n) \in D_f$, if $xI(R_i)y$ for all $i \in N'$, then $[xP(R_k)y$ for all $k \in N_1] \Rightarrow xP(R)y$ and $[yP(R_k)x$ for all $k \in N_1] \Rightarrow yP(R)x$, where $R = f(\mathbf{R})$.

We now proceed the proof of Claim 2 in two steps.

Step 1. The following statement is to be proved: For all $x, y, a, b \in X$, if $N_1 \subset N - N'$ is N' -conditionally decisive over x and y , then N_1 is N' -conditionally decisive over a and b .

Let $x, y, a, b \in X$ and let $N_1 \subset N - N'$ be N' -conditionally decisive over x and y . If $\{x, y\} = \{a, b\}$, then N_1 is N' -conditionally decisive over a and b . Consider now that $\{x, y\}$ and $\{a, b\}$ have one alternative in common. Assume without loss of generality that $y = a$. Consider $x, y = a, b$ and the profile $\mathbf{R} = (R_1, \dots, R_n) \in D_f$ such that: [for all $i \in N'$, $xI(R_i)yI(R_i)b$], [$bP(R_k)xP(R_k)y$ for all $k \in N_1$], and [$bP(R_h)x$ for all $h \in N - N' - N_1$]. Then, $bP(R)x$ follows from SPP, $xP(R)y$ follows from the assumption that N_1 is N' -conditionally decisive over x and y , where $R = f(\mathbf{R})$. By transitivity of R , $bP(R)y$. Note that $bI(R_i)y$ for all $i \in N'$ and preferences over b and y for all individuals $h \in N - N' - N_1$ are not specified, and [$bP(R_k)y$ for all $k \in N_1$]. Therefore, from IIA, it must be true that, if $bI(R_i)y$ for all $i \in N'$, then [$bP(R_k)y$ for all $k \in N_1$] \Rightarrow $bP(R)y$. Similarly, it can be shown that, if $yI(R_i)b$ for all $i \in N'$, then [$yP(R_k)b$ for all $k \in N_1$] \Rightarrow $yP(R)b$. Therefore, N_1 is N' -conditionally decisive over a and b if $y = a$. In a similar fashion, it can be shown that N_1 is N' -conditionally decisive over a and b if $\{x, y\}$ and $\{a, b\}$ have no alternatives in common.

Step 2. We show the following statement: If $N_1 \subset N'_N$ is N' -conditionally decisive over x and y and N_1 contains more than one individual, then there exists a partition, N_{11} and N_{12} , of N_1 such that N_{11} and N_{12} are non-empty, mutually exclusive, and $N_{11} \cup N_{12} = N_1$, and either N_{11} is N' -conditionally decisive over x and y or N_{12} is N' -conditionally decisive over x and y .

Let N_{11} and N_{12} be a partition of N_1 such that neither is empty, $N_{11} \cap N_{12} = \emptyset$, $N_{11} \cup N_{12} = N_1$. Consider $x, y, z \in X$ and a profile $\mathbf{R} = (R_1, \dots, R_n)$ such that $xI(R_i)yI(R_i)z$ for all $i \in N'$, $xP(R_k)y$ for all $k \in N_1$, $xP(R_h)z$ for all $h \in N_{11}$, and $zP(R_j)y$ for all $j \in N_{12}$. Since N_1 is N' -conditionally decisive over x and y , we must have $xP(R)y$, where $R = f(\mathbf{R})$. By the completeness of R , either (i) zRx or (ii) $xP(R)z$. If (i) zRx , then the transitivity of R implies $zP(R)y$. Note that the preference of each of the individual in N_{11} over z and y is not specified and is not restricted and that $zP(R_j)y$ for all $j \in N_{12}$. By IIA, N_{12} is N' -conditionally decisive over y and z . From Step 1, N_{12} is N' -conditionally

decisive over x and y . If (ii) $xP(R)z$, then, by noting that the preference of each of the individual in N_{12} over z and x is not specified and is not restricted, and that $xP(R_h)z$ for all $h \in N_{11}$, from IIA, N_{11} is N' -conditionally decisive over x and z . Again, from Step 1, N_{11} is N' -conditionally decisive over x and y .

Now, to see that Claim 2 is true, we note that $N - N'$ is N' -conditionally decisive over any x and y by virtue of SSP. From Steps 1 and 2, given that N is finite, Claim 2 follows easily.

We are now ready to show that f is hierarchically dictatorial. Note that SSP implies WPP. By Arrow's impossibility theorem, there exists $d \in N$ such that for all $x, y \in X$ and all $\mathbf{R} = (R_1, \dots, R_n) \in D_f$, $xP(R_d)y \Rightarrow xP(R)y$ where $R = f(\mathbf{R})$. Without loss of generality, let $d = 1$. Clearly, individual 1 is the level-1 dictator. Let $x, y \in X$ be such that $xI(R_1)y$. If $n = 2$, then by SPP, $xR_2y \Leftrightarrow xRy$ where $R = f(\mathbf{R})$. Thus, f is hierarchically dictatorial. If $n > 2$, from Claim 2, there exists a level-2 dictator. Without loss of generality, let individual 2 be the level-2 dictator. Suppose now that $xI(R_1)y$ and $xI(R_2)y$. If $n = 3$, by SPP, it follows that f is hierarchically dictatorial. By the repeated use of Claim 2 and SPP, we can show that for all n , f is hierarchically dictatorial. ■

To conclude the paper, we discuss the domain restriction for the hierarchical dictator result to be true in relation to the domain restriction to obtain the Arrovian dictator result. The main question we want to address is the following: Is the domain restriction that gives rise to the Arrovian dictator in the presence of IIA and weak Pareto sufficient for us to obtain the hierarchical dictators in the presence of IIA and SSP?

To begin with, let φ_L be the set of all *linear* orderings. Then the following result is trivial.

Observation 3. Let $D_f = (\varphi_L)^n$. A social welfare function is hierarchically dictatorial if and only if it satisfies SPP and IIA.

Next, we focus on the domain restriction that gives rise to Arrow's impossibility result.² Let D_f^A be the domain of f such that in the presence of IIA and weak Pareto,

²For some discussions of minimum domains on which Arrow's impossibility result holds, see, for example, Fishburn and Kelly (1997) and Kelly (1994).

f is dictatorial.

Observation 4. Let $n = 2$ and $D_f = D_f^A$. A social welfare function is hierarchically dictatorial if and only if it satisfies SPP and IIA.

However, as the following counter-example will show, Observation 4 cannot be generalized to $n > 2$.

Example 5. For simplicity and clarity, we focus on $N = \{1, 2, 3\}$ and $X = \{x, y, z\}$, though our example can be easily generalized to an arbitrary X and an arbitrary N with $n > 3$. Let D_f be such that: (i) $D_f^A \subset D_f$; (ii) for all $i \in \{1, 2, 3\}$, there exist distinct $a, b \in \{x, y, z\}$ and a profile $(R_1, R_2, R_3) \in D_f$ such that $aI(R_i)b$, $aP(R_j)b$ and $bP(R_k)a$, where $j, k \in \{1, 2, 3\} - \{i\}$; (iii) for all $i \in \{1, 2, 3\}$, there exists no profile $(R_1, R_2, R_3) \in D_f$ such that $xI(R_i)yI(R_i)z$.³ From our assumption on the domain D_f , there exists a dictator. Without loss of generality, Let 1 be the dictator. Define f as follows: for all $(R_1, R_2, R_3) \in D_f$ and all $u, v \in \{x, y, z\}$,

if $uP(R_1)v$ then $uP(R)v$;

if $uI(R_1)v$, then

$uP(R)v$ if $[(uR_2v \text{ and } uP(R_3)v) \text{ or } (uP(R_2)v \text{ and } uR_3v)]$, and

$uI(R)v$ if $[(uI(R_2)v \text{ and } uI(R_3)v), \text{ or } (uP(R_2)v \text{ and } vP(R_3)u), \text{ or } (vP(R_2)u \text{ and } uP(R_3)v)]$,

where $R = f(R_1, R_2, R_3)$. In the next, we show that f is a social welfare function, satisfies SPP and IIA, and is not hierarchically dictatorial.

R is an ordering: From the construction of the social welfare function, we first note that R is reflexive and complete. Therefore, we need only to show that R is transitive. Suppose xRy and yRz . We want to show that xRz . Suppose to the contrary that zPx . Consider individual 1's preference over x and z . Given that zPx and the way R is constructed, either $zP(R_1)x$ or $zI(R_1)x$. From xRy and yRz , individual 1's preferences over x and y , and y and z must be xR_1y and

³From Observation 3, such a domain restriction is not vacuous. See also Kelly (1994) where he shows that, in order to obtain Arrow's impossibility result, the domain on which a social welfare function is defined need not contain any profile in which an individual i is pair wise indifferent between x, y and z even though the requirement (ii) is fulfilled.

yR_1z . R_1 being transitive, it then implies that xRz . Hence, it is impossible to have $zP(R_1)x$. On the other hand, when $zI(R_1)x$ holds, given xR_1y and yR_1z , by transitivity of R_1 , we obtain $xI(R_1)yI(R_1)z$, which is an impossibility due to our domain restriction. Therefore, R is transitive.

f satisfies SPP: The fact that f satisfies SPP follows from the construction of the social welfare function f directly.

f satisfies IIA: For any $a, b \in \{x, y, z\}$, consider two profiles (R_1, R_2, R_3) and $(R'_1, R'_2, R'_3) \in D_f$ such that $aR_ib \Leftrightarrow aR'_ib$ for all $i \in \{1, 2, 3\}$. Let $R = f(R_1, R_2, R_3)$ and $R' = f(R'_1, R'_2, R'_3)$. When $aP(R_1)b$ and $aP(R'_1)b$, from the construction of f , $aP(R)b$ and $aP(R')b$. When $aI(R_1)b$ and $aI(R'_1)b$, given that $aR_2b \Leftrightarrow aR'_2b$ and $aR_3b \Leftrightarrow aR'_3b$, from the construction of f , we again obtain aRb iff $aR'b$. Therefore, f satisfies IIA.

f is not hierarchically dictatorial: From the requirement (ii) of the domain D_f in this example, without loss of generality, let a profile $(R_1, R_2, R_3) \in D_f$ be such that $xI(R_1)y$, $xP(R_2)y$ and $yP(R_3)x$ and let $R = f(R_1, R_2, R_3)$. Then, according to our construction, $xI(R)y$. On the other hand, if f were hierarchically dictatorial, given $xI(R_1)y$ and individual 1 being the dictator in the sense of Arrow, since both individuals 2 and 3 have strict preferences over x and y , the social preference over x and y according to a hierarchically dictatorial social welfare function would have to be strict. Therefore, f is not hierarchically dictatorial.

■

The message of the above counter-example is simple. In general, the domain restriction for obtaining Arrow's dictatorial result in the presence of IIA and weak Pareto is not sufficient to obtain the result on hierarchical dictators in the presence of IIA and strong Pareto. To present an equivalent domain restriction for the two types of results, we consider the following.

Let Ω be any set of preference orderings such that for every $k = 2, \dots, n$, any social welfare function on Ω^k satisfying IIA and weak Pareto is dictatorial.

Theorem 6. Let D_f be such that: (i) $\Omega^n \subset D_f$; (ii) if for all $i \in N$ and all distinct $x, y \in X$ such that $xI(R_i^*)y$ for some profile $(R_1^*, \dots, R_i^*, \dots, R_n^*) \in D_f$, then for all

$z \in X - \{x, y\}$ there exists a profile $(R_1, \dots, R_n) \in D_f$ such that $xI(R_i)yI(R_i)z$.⁴ Then, a social welfare function on D_f is hierarchically dictatorial if and only if it satisfies SPP and IIA.

Proof. Let D_f be as specified in Theorem 6. It is straightforward to check that if a social welfare function on D_f is hierarchically dictatorial then it satisfies SPP and IIA. We now show that if a social welfare function on D_f satisfies SPP and IIA, then it is hierarchically dictatorial.

Let f be a social welfare function on D_f satisfying SPP and IIA. Note that SPP implies weak Pareto. From our domain assumption, there exists a dictator $d \in N$: for all $x, y \in X$ and all $\mathbf{R} = (R_1, \dots, R_n) \in D_f$, $xP(R_d)y \Rightarrow xP(R)y$ where $R = f(\mathbf{R})$. Without loss of generality, let $d = 1$. Apparently, 1 is the level-1 dictator. Suppose now that for some distinct $x, y \in X$ and a profile $\mathbf{R}^* = (R_1^*, \dots, R_n^*) \in D_f$, $xI(R_1^*)y$. Then, from our domain restriction, for all $z \in X - \{x, y\}$, there exists a profile $\mathbf{R}' = (R'_1, \dots, R'_n) \in D_f$ such that $xI(R'_1)yI(R'_1)z$. By IIA and SPP, and from our assumption on the domain D_f , there exists $d_2 \in N - \{1\}$ such that, for all $a, b \in X$ and all $\mathbf{R} = (R_1, \dots, R_n) \in D_f$, if $aI(R_1)b$ then $aP(R_{d_2})b \Rightarrow aP(R)b$, where $R = f(\mathbf{R})$. Without loss of generality, let $d_2 = 2$. Apparently, 2 is the level-2 dictator. Similarly, we can show that, if there are cases in which both individuals 1 and 2 are indifferent between two alternatives, then there exists $d_3 \in N - \{1, 2\}$ such that d_3 is the level-3 dictator. By repeating the above process if necessary, we can show that f is hierarchically dictatorial. ■

The message of our results seems to suggest the following. When a social welfare function f is defined on either $D_f = (\varphi_L)^n$ or $D_f = \varphi^n$, then, f must be dictatorial if it satisfies IIA and weak Pareto, and, at the same time, f must be hierarchically dictatorial if it satisfies IIA and strong Pareto. In such cases, the domain restrictions for obtaining a dictatorial social welfare function in the presence of IIA and weak Pareto are identical to those for obtaining a hierarchically dictatorial social welfare function when IIA and strong Pareto are satisfied. However, once more profiles are added to $(\varphi_L)^n$ to obtain the domain D_f on which a social welfare function f is

⁴It may be of interest to note that, given the transitivity of indifference relations, the requirement (ii) is equivalent to the following: if for all $i \in N$, there exist distinct $a, b \in X$ such that $aI(R_i^*)b$ for some profile $(R_1^*, \dots, R_n^*) \in D_f$, then for all distinct $x, y, z \in X$ there exists a profile $(R_1, \dots, R_n) \in D_f$ such that $xI(R_i)yI(R_i)z$.

defined, even if f is dictatorial when IIA and weak Pareto are present, f defined on the same D_f may not be hierarchically dictatorial in the presence of IIA and strong Pareto: this can happen if some profiles containing indifferences between and among alternatives are added, and, at the same time, the addition leaves at least two distinct alternatives x and y that are indifferent not to find any profile and any third alternative z such that x , y and z are pair wise indifferent under this profile. Therefore, to obtain the same domain restriction for the dictator result and for the hierarchical dictator result, the addition of profiles containing indifference relations must be “enough” so that whenever the domain contains a profile in which some two alternatives are indifferent according to an individual’s preference ordering, profiles that give pair wise indifference relations for any three alternatives according to this individual’s preference ordering must be added as well. This restriction is in sharp contrast to the domain restriction that gives rise to a dictatorial social welfare function when IIA and weak Pareto are present, as Kelly (1994) shows that, in order to have Arrow’s impossibility result, it is not necessary for a domain on which a social welfare function satisfying IIA and weak Pareto is defined to include such profiles.

References

- Arrow, K. J. (1951). *Social Choice and Individual Values*, New York: Wiley. Second Edition, 1963.
- Craven, J. (1992). *Social Choice: A Framework for Collective Decisions and Individual Judgements*, Cambridge: Cambridge University Press.
- Fishburn, P. C. and J. S. Kelly (1997). Super-Arrovian domains with strict preferences. *SIAM Journal of Discrete Mathematics* 10, 83-95.
- Kelly, J. S. (1994). The free triple assumption. *Social Choice and Welfare* 11, 97-101.