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Abstract. This paper studies measures of group deprivation in a multi-dimensional framework where, for each individual and each attribute under consideration, the only information that one uses is whether the individual is deprived in terms of that attribute. We consider two classes of measures; one class needs micro-level data while the other class deals with aggregate data. We focus on the axiomatic properties of these two classes.

JEL Classification Numbers: D0, D63, I32

1. Introduction

This paper studies the structures of two classes of simple measures of overall deprivation of a group (or, equivalently, a society) in a multi-attribute or multi-dimensional framework where, for each individual in the group and each attribute, there are exactly two levels of deprivation of the individual in terms of that attribute, namely, 0 (indicating that the individual is not deprived in terms the attribute under consideration) and 1 (indicating that the individual is deprived in terms of the attribute under consideration). The informal basis of our group deprivation measures is, therefore, an $n \times m$ matrix where n is the number of individuals in the group, m is the number of attributes, and, for every individual and every attribute, the corresponding entry in the matrix is either 0 or 1. The problem is one of aggregating such a deprivation matrix to reach a single index which reflects the overall deprivation of the group. This contrasts strongly with the framework used in many other contributions (see, for instance, Alkire and Foster (2011), Bourguignon and Chakravarty (2003), Chakravarty and Silber (2007), Dutta, Pattanaik and Xu (2003), and Tsui (2002)) where each attribute is assumed to be cardinally measurable along a real interval. The assumption allows one to use, for each attribute, a cardinal measure of an individual's deprivation in terms of the attribute; the measure is simply the shortfall, if any, of the individual's achieved level for the attribute from a pre-specified benchmark[1], normalized through division by the benchmark level. It is clear that our framework is informationally much less rich than the more usual cardinal framework. In some ways, this informational parsimony may be an advantage. Many important attributes, such as health, overall nourishment, education, and housing do not lend themselves very well to cardinal measurement. Often the reason is that the attribute under consideration is itself multi-dimensional, and, though each of the dimensions may yield to cardinal measurement, there may not be general agreement about how to aggregate over these different dimensions so as to have an overall cardinal measure of the attribute itself. Consider health. There are numerous dimensions of a person's health - the person's weight, her level of energy, frequency of her illnesses, her blood pressure, the level of her cholesterol, and so on. Many of these separate dimensions are amenable to cardinal measurement, but it is extremely difficult to provide a cardinal measurement of an individual's overall health. Nor is it practicable to include in one's analysis all these different dimensions of health as separate attributes on their own. The same seems to be true for the attributes of nourishment, housing, and even education. It is not our intention to claim that cardinal measurement is never possible for important attributes. In principle, leisure, seen as hours outside work, seems to be cardinally measurable. Similarly, if, in considering nourishment, we focus on, say, calorie consumption as an exclusive "proxy" for nourishment, then we shall have cardinal measurability for nourishment, though such measurability would be secured at the cost of the narrowness of our interpretation of nourishment. What we would like to emphasize is that the assumption of cardinal measurement is demanding and unlikely to be satisfied for many crucial dimensions of deprivation. Even ordinal distinctions between

[1] If the individual's achievement is not less than the benchmark, her deprivation is considered to be 0 for the attribute under consideration.

different levels of health, such as splendid health, reasonable health, poor health, very poor health, which are often a part of our everyday language, may run into problems since people often differ in their assessment of whether a person's health is reasonable or poor and whether a person's health is poor or very poor. What can be done given these difficulties involved in the measurement of attributes? One possibility may be to relax one's requirements with respect to the measurement of an individual's deprivation along different dimensions. In this paper, we work with a coarse form of ordinal measurement of a person's deprivation along a given dimension, where there are exactly two levels of deprivation of the individual: either the individual is deprived along the dimension under consideration or she is not. Our paper seeks to explore how far one can go with such a minimal form of measurement of an individual's deprivation along every dimension.

In this paper, we study two classes of simple measures for aggregating deprivation matrices to overall deprivation measures for the society. For a given deprivation matrix,

- intuitively, a member of our first class of measures computes the overall deprivation of the group through the following four steps:
  - Step 1.1: for each of the identified attributes, a weight is assigned; these weights reflect the society's view of the relative importance of various attributes and can be different across different attributes;
  - Step 1.2: for each individual and using the set of weights chosen in Step 1.1, a weighted sum of the individual's deprivations along all the dimensions is calculated; this simple weighted sum may be called the individual's *nominal deprivation level*;
  - Step 1.3: a suitably chosen function is applied to transform each individual's nominal deprivation level to another number which may be called the individual's *real deprivation level*;
  - Step 1.4: the society's overall deprivation is then obtained by summing up the real deprivation levels obtained in Step 1.3.
- a member of our second class of measures computes overall group deprivation by following the following two steps:
  - Step 2.1: for each attribute, one counts the number of individuals who are deprived along that attribute;
  - Step 2.2: a suitably chosen, multi-variate function is applied to transform the numbers obtained in Step 2.1 to derive a measure of the society's overall deprivation.

It may be noted that, measures in the first class conform to the intuition underlying conventional welfare economics: the society's overall deprivation is conceptually the result of aggregating the overall deprivations of all the individuals in the society. Measures in the second class lack such intuition. On the other hand, measures in the first class have stringent requirements for the data to be gathered: to compute the society's overall deprivation using a measure belonging to this class, one needs to gather individual or micro level data. In contrast, measures in the second class have less stringent requirements with respect to data: to compute the society's overall deprivation using a measure in our second class, it is sufficient to have aggregate data, i.e., it is enough to know, for each attribute, the percentage of the population who are deprived in terms of that attribute.[2]

Our study of the two classes of measures focuses on understanding their structures. Following an established tradition in the literature, we use an axiomatic approach to examine what properties characterize these two classes of measures. The exercise is useful in identifying the attractive as well as unattractive features of the measures in each class.[3]

It may be noted that, some special cases of the two classes of measures have figured prominently in both theoretical and applied works on multi dimensional deprivation, see, among others, Alkire and Foster (2011), Bossert, Chakravarty, and D'Ambrosio (2013), Ruiz (2011), UNDP (various years), Whelan, Nolan, and Maitre (2012).

The remainder of the paper is organized as follows. In Section 2, we introduce some basic notation and definitions needed for our analysis. Section 3 presents the axioms and the results. We conclude in Section 4.

2. Notation and Definitions

Let $N = \{1, \cdots, n\}$ be a given finite set of individuals with $n \geq 2$ and let $F = \{f_1, \cdots, f_m\}$ be a finite set of attributes with $m \geq 2$; we shall refer to N as the group or society under consideration. Let $M = \{1, \cdots, m\}$. Let $\mathcal{D}$ be the class of all $n \times m$ matrices, D, such that each entry in D is either 0 or 1.The elements of $\mathcal{D}$ will be denoted by $C \equiv (c_{ij})_{n\times m}, D = (d_{ij})_{n\times m}$,etc., and will be called deprivation matrices. For all $D \in \mathcal{D}$, all $i \in N$ and all $j \in M$, the entry d_{ij} in the deprivation matrix D will be interpreted as i's level of deprivation in terms of attribute f_j : if $d_{ij} = 0$, then i is not deprived in terms of attribute f_j in the matrix D; on the other hand, if $d_{ij} = 1$, then i is deprived in terms of attribute f_j in deprivation matrix D.For each $i \in N$, let $d_{i\bullet} = (d_{i1}, \cdots, d_{I'm})$ denote the deprivation status of individual i, in D, along the m dimensions. Similarly, for each $j \in M$, let $d_{\bullet j} = (d_{1j}, \cdots, d_{nj})$ denote the vector of the n individuals' deprivation levels, in D, in terms of the given attribute f_j.

Let $h : \mathcal{D} \to [0, 1]$ be a function from $\mathcal{D}$ to the closed interval $[0, 1]$. The interpretation of the function h is as follows: for all $C_{n\times m}, D_{n\times m} \in \mathcal{D}, h(C) \geq h(D)$ is interpreted as the degree of deprivation that the society has under C is at least as high as the degree of deprivation under D, $h(C) > h(D)$ indicates that the degree of deprivation that the society has under C is higher than the degree of deprivation under D, and $h(C) = h(D)$ is interpreted as the degree of deprivation that the society has under C is exactly as high as the degree of deprivation under D.

For any $C = (c_{ij})_{n\times m},\ D = (d_{ij})_{n\times m} \in \mathcal{D}$, any $p \in N$ and any $k \in M$, we say that, (i) C and D are (pk)-variant if $c_{pk} \neq d_{pk}$ and $c_{ij} = d_{ij}$ for all $ij \neq pk$; that is, C and D

[2]See Dutta, Pattanaik and Xu (2003), and Pattanaik, Reddy and Xu (2012) for further discussions on related issues.

[3]In a similar framework to ours, Bossert, Chakravarty, and D'Ambrosio (2013) present an axiomatic study of a very specific measure which is defined as the simple sum of all the individuals' nominal deprivation levels.

are identical except the pk-th elements, and (ii) C and D are (pk)-invariant if $c_{pk} = d_{pk}$; that is, C and D have the same pk-th elements. A deprivation matrix $D = (d_{ij})_{n\times m} \in \mathcal{D}$ is said to be a simple deprivation matrix if, for some $p \in N$, $d_{i\bullet}$ is the zero vector for all $i \in N\backslash\{p\}$; that is, D is such that all but possibly at most one individual are not deprived in any of the m attributes.

Let $x \in \{0,1\}^m$, and $C(i,x) \in \mathcal{D}$ be a simple deprivation matrix such that $c_{i\bullet} = x$ for some $i \in N$. For all $x, y \in \{0,1\}^m$, let $x \succeq y$ if $h(C(i,x)) \geq h(C(i,y))$ for all $i \in N$ and $x \succ y$ if $h(C(i,x)) > h(C(i,y))$ for all $i \in N$.

For any two deprivation matrices C and D, we say that C dominates D, to be denoted by $C \succ^{dom} D$, if, (i) for each $j \in M$, there is a permutation σ^j over N such that $c_{ij} = d_{\sigma^j(i)j}$ for all $i \in N$, and (ii) for some permutation π over N, $[c_{i\bullet} \succeq d_{\pi(i)\bullet}$ for all $i \in N]$ and $[c_{i\bullet} \succ d_{\pi(i)\bullet}$ for some $i \in N]$.

3. Measures and Their Axiomatic Derivations

In this Section, we study two classes of measures of overall deprivation defined as follows axiomatically: for all $C = (c_{ij})_{n\times m} \in \mathcal{D}$

(First Class):

$$h(C) = \sum_{i\in N} g(\sum_{j\in M} \omega_j c_{ij})/n$$

where $\omega_1 > 0, \cdots, \omega_m > 0$, $\sum_{j=1}^m \omega_j = 1$, $g : [0,1] \to [0,1]$, and $g(0) = 0$ and $g(1) = 1$;

(Second Class):

$$h(C) = \xi(\sum_{i\in N} c_{i1}, \cdots, \sum_{i\in N} c_{ij}, \cdots, \sum_{i\in N} c_{im})$$

where ξ is an increasing function $\xi : \{0,1,\cdots,n\}^m \to [0,\infty)$ with $\xi(0,\cdots,0) = 0$ and $\xi(1,\cdots,1) = 1$.

3.1. Axioms.

We begin our study by noting the following properties, some of which are to be imposed on a measure of the society's overall deprivation.

- **Normalization.** For all $D = (d_{ij})_{n\times m}$, if $d_{ij} = \delta \in \{0,1\}$ for all $i \in N$ and all $j \in M$, then $h(D) = \delta$.
- **Anonymity.** Let σ be a bijection from N to N. Then, for all $C, D \in \mathcal{D}$, if $c_{i\bullet} = d_{\sigma(i)\bullet}$ for all $i \in N$, then $h(C) = h(D)$.
- **Monotonicity.** For all $C = (c_{ij})_{n\times m}$, $D = (d_{ij})_{n\times m}$, if $c_{ij} \geq d_{ij}$ for all $i \in N$ and all $j \in M$, and $C \neq D$, then $h(C) > h(D)$.
- **Independence.** For all $C, D, C', D' \in \mathcal{D}$, and for all $k \in N$, if $[(c_{i\bullet} = d_{i\bullet}$ and $c'_{i\bullet} = d'_{i\bullet}$ for all $i \in N\backslash\{k\}$ and $(c_{k\bullet} = c'_{k\bullet}, d_{k\bullet} = d'_{k\bullet})]$, then $h(C) - h(D) = h(C') - h(D')$.
- **Additivity(I).** For all simple deprivation matrices, $C, D, C', D' \in \mathcal{D}$, and for all $p \in N$ and all $q \in M$, if $c_{pq} = d_{pq} = 1$, $c'_{pq} = d'_{pq} = 0$, C and C' are (pq)-variant, D and D' are (pq)-variant, then $h(C) \geq h(D) \Leftrightarrow h(C') \geq h(D')$.

- **Additivity (II).** $\forall C, D, C', D' \in \mathcal{D}$, $\forall p \in N$, $q \in M$, if $c_{pq} = d_{pq} = 1$, $c'_{pq} = d'_{pq} = 0$, C and C' are (pq)-variant, and D and D' are (pq)-variant, then $C \succ^{dom} D \Rightarrow C' \succ^{dom} D'$.
- **Strong Additivity (I)** $\forall C, D, C', D' \in \mathcal{D}$, $\forall p \in N$, $q \in M$, if $c_{pq} = d_{pq} = 1$, $c'_{pq} = d'_{pq} = 0$, C and C' are (pq)-variant, and D and D' are (pq)-variant, then $h(C) \geq h(D) \Leftrightarrow h(C') \geq h(D')$.

Normalization is straightforward: if no one in the group N is deprived along any dimension, then the overall deprivation index for N is 0, and if everyone in N is deprived along every dimension, then the overall deprivation index for N is 1. Anonymity requires that the interchange of any two rows of a deprivation matrix does not affect the overall deprivation. It essentially says that the name of an individual has no significance in measuring overall deprivation of the society. It may be noted that in the literature on measuring multi-dimensional deprivation, Anonymity is also called Symmetry. Monotonicity requires that, if every individual under C is as at least deprived as under D and some individual under C is deprived while the same individual is non-deprived under D, then the overall deprivation level under C is higher than that under D. These three properties are fairly standard in the literature on multi dimensional approach to deprivation, see, among others, Bourguignon and Chakravarty (2003), and Tsui (2002).

Intuitively, Independence requires that the overall deprivation measure is separable with respect to individuals' deprivations: if two deprivation matrices differ only with respect to a single individual's deprivations along the m dimensions, then the difference between the overall deprivations under the two deprivation matrices is independent of all other individuals' deprivations. This property has its root in the literature on measuring deprivation in a uni-dimensional framework, see, for example, Chakraborty, Pattanaik and Xu (2008). In a multi dimensional framework, it has often been invoked in the discussion of the row-first, column-second procedures for measuring overall deprivation, see, for example, Dutta, Pattanaik and Xu (2003), Pattanaik, Reddy and Xu (2012), and Tsui (2002).[4]

The basic idea underlying the three additivity-type axioms, Additivity (I), Additivity (II) and Strong Additivity (I), is fairly simple and intuitive, though they differ with respect to the applicability of this basic idea to various deprivation situations. The idea boils down to how the comparison of two deprivation matrices, C and D, changes when the same individual along a given dimension under C and D switches from the same status of being deprived (resp. being non-deprived) to the same status of being non-deprived (resp. being deprived): it requires that the comparison of C and D after such change be analogous to the comparison of C and D before such change. Additivity (I) confines the comparisons to *simple deprivation matrices*, Additivity (II) applies the comparisons to deprivation matrices that have the *dominance* relation, while Strong Additivity (I) imposes no restriction on deprivation matrices. It may be noted that Strong Additivity (I), introduced in Pattanaik, Reddy and Xu (2012), is logically stronger than Additivity (I), but there is no such logical

[4]A row-first, column-second procedure corresponds to the procedure that first aggregates each individual's deprivations along the identified dimensions into an deprivation levels, and then the overall deprivation for the society is obtained by aggregating those individual deprivation levels.

relation between Additivity (I) and Additivity (II), and between Additivity (II) and Strong Additivity (I) since Additivity (II) is concerned with how the dominance relation $\succeq^{dom}$ between the two matrices is being preserved, while Additivity (I) and Additivity (II) are concerned with how the ranking of the two matrices in terms of the measure h is being preserved.

3.2. **Characterization of the first class of measures.** In this subsection, we study the structure of the first class of measures. To begin with, we present the following result which shows that the overall deprivation of the society is the sum of individuals' overall deprivations along the m dimensions if certain axioms are imposed on the overall deprivation of the society.

Proposition 1. *A deprivation measure h satisfies Normalization, Monotonicity, Anonymity and Independence if and only if there exists an increasing function $\varphi : \{0,1\}^m \to [0,1]$ with $\varphi(0,\cdots,0) = 0$, $\varphi(1,\cdots,1) = 1/n$ such that, for all $C = (c_{ij})_{n\times m} \in \mathcal{D}$, $h(C) = \sum_{i\in N} \varphi(c_{i\bullet})$.*

Proof. It may be checked that, each measure of the class (add.c), $h(C) = \sum_{i\in N} \varphi(c_{i\bullet})$, where $\varphi : \{0,1\}^m \to [0,1]$ is an increasing function, satisfies Normalization, Monotonicity, Anonymity and Independence. Therefore, we need only to show that, if a measure h satisfies Normalization, Monotonicity, Anonymity and Independence, then there exists an increasing function $\varphi : \{0,1\}^m \to [0,1]$ such that, for all $C = (c_{ij})_{n\times m} \in \mathcal{D}$, $h(C) = \sum_{i\in N} \varphi(c_{i\bullet})$.

Let h satisfy Normalization, Monotonicity, Anonymity and Independence, and let $C = (c_{ij})_{n\times m} \in \mathcal{D}$. Consider

$$h\begin{pmatrix} c_{1\bullet} \\ c_{2\bullet} \\ \vdots \\ c_{n\bullet} \end{pmatrix} - h\begin{pmatrix} 0_{1\bullet} \\ c_{2\bullet} \\ \vdots \\ c_{n\bullet} \end{pmatrix} \text{ and } h\begin{pmatrix} c_{1\bullet} \\ 0_{2\bullet} \\ \vdots \\ 0_{n\bullet} \end{pmatrix} - h\begin{pmatrix} 0_{1\bullet} \\ 0_{2\bullet} \\ \vdots \\ 0_{n\bullet} \end{pmatrix}$$

By Normalization,

$$h\begin{pmatrix} 0_{1\bullet} \\ 0_{2\bullet} \\ \vdots \\ 0_{n\bullet} \end{pmatrix} = 0$$

and by Independence

$$h\begin{pmatrix} c_{1\bullet} \\ c_{2\bullet} \\ \vdots \\ c_{n\bullet} \end{pmatrix} - h\begin{pmatrix} 0_{1\bullet} \\ c_{2\bullet} \\ \vdots \\ c_{n\bullet} \end{pmatrix} = h\begin{pmatrix} c_{1\bullet} \\ 0_{2\bullet} \\ \vdots \\ 0_{n\bullet} \end{pmatrix} - h\begin{pmatrix} 0_{1\bullet} \\ 0_{2\bullet} \\ \vdots \\ 0_{n\bullet} \end{pmatrix}$$

Therefore,

$$h\begin{pmatrix} c_{1\bullet} \\ c_{2\bullet} \\ \vdots \\ c_{n\bullet} \end{pmatrix} = h\begin{pmatrix} 0_{1\bullet} \\ c_{2\bullet} \\ \vdots \\ c_{n\bullet} \end{pmatrix} + h\begin{pmatrix} c_{1\bullet} \\ 0_{2\bullet} \\ \vdots \\ 0_{n\bullet} \end{pmatrix}.$$

Similarly, it can be shown that, from Independence and Normalization,

$$h\begin{pmatrix} 0_{1\bullet} \\ c_{2\bullet} \\ \vdots \\ c_{n\bullet} \end{pmatrix} = h\begin{pmatrix} 0_{1\bullet} \\ c_{2\bullet} \\ c_{3\bullet} \\ \vdots \\ c_{n\bullet} \end{pmatrix} + h\begin{pmatrix} 0_{1\bullet} \\ c_{2\bullet} \\ 0_{3\bullet} \\ \vdots \\ 0_{n\bullet} \end{pmatrix}.$$

By the repeated use of the above method and from Independence and Normalization, we have the following

$$h\begin{pmatrix} c_{1\bullet} \\ c_{2\bullet} \\ \vdots \\ c_{n\bullet} \end{pmatrix} = h\begin{pmatrix} c_{1\bullet} \\ 0_{2\bullet} \\ 0_{3\bullet} \\ \vdots \\ 0_{n\bullet} \end{pmatrix} + h\begin{pmatrix} 0_{1\bullet} \\ c_{2\bullet} \\ 0_{3\bullet} \\ \vdots \\ 0_{n\bullet} \end{pmatrix} + \cdots + h\begin{pmatrix} 0_{1\bullet} \\ 0_{2\bullet} \\ \vdots \\ 0_{n-1\bullet} \\ c_{n\bullet} \end{pmatrix}$$

For each $i \in N$, let $\varphi_i(c_{i\bullet}) = h(C_i)$ where C_i is the deprivation matrix in which the deprivation vector of individual i is given by $c_{i\bullet}$ and all other individuals' deprivation vectors are zero vectors. Since the choice of $c_{i\bullet}$ is arbitrary, each φ_i is thus a function due to that h is a function. From Normalization, it is clear that $0 \leq \varphi_i(c_{i\bullet}) < 1$ and $\varphi_i(c_{i\bullet}) = 0$ when $c_{i\bullet}$ is the 0 vector. By Anonymity, φ_is are the same and let $\varphi_i = \varphi$ for all $i \in N$. By Monotonicity, φ is increasing. By Normalization, $\varphi(1_{i\bullet}) = 1/n$.

We have therefore shown that, for the above defined function φ, for all $C = (c_{ij})_{n\times m} \in \mathcal{D}$, $h(C) = \sum_{i\in N} \varphi(c_{i\bullet})$. □

Therefore, the combination of Normalization, Anonymity, Monotonicity and Independence to be imposed on a deprivation measure h implies that h is additive across individuals. Our next result establishes that the φ function figured in Proposition 1 is additive when the number of attributes is no greater than 4 if we are willing to require h to satisfy Additivity (I).

Proposition 2. *Suppose $m \leq 4$. A deprivation measure h satisfies Normalization, Monotonicity, Anonymity, Independence and Additivity (I) if and only if there exist an increasing function $g : [0,1] \to [0,\infty)$ with $g(0) = 0$ and $g(1) = 1/n$, and constants $\omega_1 > 0, \cdots, \omega_m > 0$ with $\omega_1 + \cdots + \omega_m = 1$ such that, and for all $C = (c_{ij})_{n\times m} \in \mathcal{D}$, $h(C) = \sum_{i\in N} g(\sum_{j=1}^{m} \omega_j c_{ij})$.*

Proof. Let a deprivation measure h satisfy Normalization, Monotonicity, Anonymity, Independence and Additivity (I). From Proposition 1, there exists an increasing function $\varphi : \{0,1\}^m \to [0,1]$ with $\varphi(0,\cdots,0) = 0$ and $\varphi(1,\cdots,1) = 1/n$ such that,

$$\text{for all } C = (c_{ij})_{n\times m} \in \mathcal{D},\ h(C) = \sum_{i\in N} \varphi(c_{i\bullet}) \tag{1}$$

Therefore, it suffices to show that φ is given by $\varphi(c_{i\bullet}) = g(\sum_{j=1}^m \omega_j c_{ij})$ for some increasing function $g : [0,1] \to [0,\infty)$ with $g(0) = 0$ and $g(1) = 1/n$, and constants $\omega_1 > 0,\cdots,\omega_m > 0$ such that, $\omega_1 + \cdots + \omega_m = 1$.

Let $i \in N$, and $c_{i\bullet}, c'_{i\bullet}, d_{i\bullet}, d'_{i\bullet} \in \{0,1\}^m$ be such that $c_{i\bullet} - c'_{i\bullet} = d_{i\bullet} - d'_{i\bullet} = (1_j; 0_{-j}) \in \{0,1\}^m$ for some $j \in M$; that is, $c_{i\bullet}$ and $c'_{i\bullet}$ are identical except at the jth component, and $d_{i\bullet}$ and $d'_{i\bullet}$ are identical except at the jth component. Consider the following simple deprivation matrices, $C, C', D, D' \in \mathcal{D}$ such as, the ith rows of C, C', D, D' are given respectively by $c_{i\bullet}, c'_{i\bullet}, d_{i\bullet}, d'_{i\bullet}$, and each of the other rows of each deprivation matrix is the zero vector. Then C and C' are (ij)-variant, and D and D' are (ij)-variant. By Additivity (I), we have

$$h(C) \geq h(D) \Leftrightarrow h(C') \geq h(D') \tag{2}$$

Then, from (1), we have

$$\varphi(c_{i\bullet}) \geq \varphi(d_{i\bullet}) \text{ iff } \varphi(c'_{i\bullet}) \geq \varphi(d'_{i\bullet}) \tag{3}$$

In other words, we have shown that the function φ satisfies the following property:

$$\begin{gathered}\text{For all } x, y, a, b \in \{0,1\}^m \text{ and for all } j \in M = \{1,\cdots,m\},\\ \text{if } x - a = y - b = (1_j, 0_{-j}) \in \{0,1\}^m,\ \text{then } \varphi(a) \geq \varphi(b) \text{ iff } \varphi(x) \geq \varphi(y)\end{gathered} \tag{4}$$

Note that $m \leq 4$. Then, there exist $\alpha_1,\cdots,\alpha_m$, and for all $c_{i\bullet}, d_{i\bullet} \in \{0,1\}^m$, $\varphi(c_{i\bullet}) \geq \varphi(d_{i\bullet}) \Leftrightarrow \sum_{j=1}^m \alpha_j c_{ij} \geq \sum_{j=1}^m \alpha_j d_{ij}$ (see Kraft, Pratt and Seidenberg (1959), or Fishburn (1996)). Since $\varphi(c_{i\bullet})$ is increasing in each of its argument, $\alpha_j > 0$ for all $j \in M$. Let $\omega_j = \alpha_j / \sum_{k=1}^m \alpha_k$. Then, $\omega_1 > 0,\cdots,\omega_m > 0$ and $\omega_1 + \cdots + \omega_m = 1$. Since φ is a function and given that $\varphi(c_{i\bullet}) \geq \varphi(d_{i\bullet}) \Leftrightarrow \sum_{j=1}^m \omega_j c_{ij} \geq \sum_{j=1}^m \omega_j d_{ij}$ for all $c_{i\bullet}, d_{i\bullet} \in \{0,1\}^m$, there exists an increasing function $g : [0,1] \to [0,\infty)$ such that $\varphi(c_{i\bullet}) = g(\sum_{j=1}^m \omega_j c_{ij})$. Note that $\varphi(0_{i\bullet}) = 0$ and $\varphi(1_{i\bullet}) = 1/n$ by Normalization. We then have $g(0) = 0$ and $g(1) = 1/n$ □

It may be noted the result in Proposition 2 holds for $m \leq 4$. When $m \geq 5$, Additivity (I) is not sufficient to guarantee that the φ function in Proposition 1 is additive. To see this, we consider the following example, which is modified from Kraft, Pratt and Seidenberg (1959).

Example 1. For simplicity, we focus on the φ function figured in Proposition 1. Consider $m = 5$ and a φ function given below:

$$\begin{aligned}\varphi(0,1,1,0,1) &> \varphi(1,0,0,1,0),\\ \varphi(1,0,0,0,1) &> \varphi(0,1,1,0,0),\\ \varphi(0,0,1,1,0) &> \varphi(0,1,0,0,1),\\ \varphi(0,1,0,0,0) &> \varphi(0,0,1,0,1).\end{aligned}$$

It can be checked that this φ satisfies the corresponding Weak Additivity property (4) stated in the proof of Proposition 2. If the result of Proposition 2 holds, then, we would have, for some $\omega_1 > 0, \omega_2 > 0, \omega_3 > 0, \omega_4 > 0, \omega_5 > 0$:

$$\begin{aligned}\omega_2+\omega_3+\omega_5 &> \omega_1+\omega_4,\\ \omega_1+\omega_5 &> \omega_2+\omega_3,\\ \omega_3+\omega_4 &> \omega_2+\omega_5,\\ \omega_2 &> \omega_3+\omega_5.\end{aligned}$$

Note that, from the above inequalities, we have

$$\omega_2+\omega_3+\omega_5+\omega_1+\omega_5+\omega_3+\omega_4+\omega_2 > \omega_1+\omega_4+\omega_2+\omega_3+\omega_2+\omega_5+\omega_3+\omega_5$$

or

$$\omega_1+2\omega_2+2\omega_3+\omega_4+2\omega_5 > \omega_1+2\omega_2+2\omega_3+\omega_4+2\omega_5$$

a contradiction.

The above example suggests that, in search for an additive measure, Additivity (I) needs to be replaced. It turns out that if Additivity (I) is replaced by Additivity (II), then we can obtain the result for $m \geq 5$, as reported in Proposition 3.

Proposition 3. *Suppose that n is large relative to m. A deprivation measure h satisfies Normalization, Monotonicity, Anonymity, Independence and Additivity (II) if and only if, there exist an increasing function $g : [0,1] \to [0,\infty)$ with $g(0) = 0$ and $g(1) = 1/n$, and constants $\omega_1 > 0, \cdots, \omega_m > 0$ such that, $\omega_1 + \cdots + \omega_m = 1$, and for all $C = (c_{ij})_{n\times m} \in \mathcal{D}$, $h(C) = \sum_{i\in N} g(\sum_{j=1}^m \omega_j c_{ij})$*

Proof. Let a deprivation measure h satisfy Normalization, Monotonicity, Anonymity, Independence and Additivity (II). From Proposition 1, there exists an increasing function $\varphi : \{0,1\}^m \to [0,1]$ such that,

$$\begin{aligned}&\text{for all } C = (c_{ij})_{n\times m} \in \mathcal{D},\ h(C) = \sum_{i\in N} \varphi(c_{i\bullet})\\ &\text{with } \varphi(0,\cdots,0) = 0 \text{ and } \varphi(1,\cdots,1) = 1/n\end{aligned} \tag{5}$$

Therefore, it suffices to show that φ is given by $\varphi(c_{i\bullet}) = g(\sum_{j=1}^m \omega_j c_{ij})$ for some increasing function $g : [0,1] \to [0,\infty)$ with $g(0) = 0$ and $g(1) = 1/n$, and constants $\omega_1 > 0, \cdots, \omega_m > 0$ such that, $\omega_1 + \cdots + \omega_m = 1$.

We first show the following:

$$\text{(6)}\quad \begin{aligned}&\text{For all integer } k \geq 2, \text{ and all } x^p, y^p \in \{0,1\}^m \text{ with } p = 1, \cdots, k,\\ &\text{and } (\forall j \in M : |\{j : x^p_j = 1, p = 1, \cdots, k\}| = |\{j : y^p_j = 1, p = 1, \cdots, k\}|),\\ &\text{if } \varphi(x^p) \geq \varphi(y^p) \text{ for all } p < k, \text{ then } \varphi(y^k) \geq \varphi(x^k)\end{aligned}$$

Suppose to the contrary that (6) does not hold; that is,

$$\text{(7)}\quad \begin{aligned}&\text{for some } k \geq 2, \text{ there are } x^p, y^p \in \{0,1\}^m \text{ with } p = 1, \cdots, k \text{ such that}\\ &(\forall j \in M : |\{j : x^p_j = 1, p = 1, \cdots, k\}| = |\{j : y^p_j = 1, p = 1, \cdots, k\}|),\\ &\varphi(x^p) \geq \varphi(y^p) \text{ for all } p \in \{1, \cdots, k\}, \text{ and } \varphi(x^p) > \varphi(y^p) \text{ for some } p \in \{1, \cdots, k\}\end{aligned}$$

From (5) and the definition of $\succeq$, we have $x^p \succeq y^p$ for all $p = 1, \cdots, k$ and $x^p \succ y^p$ for some $p \in \{1, \cdots, k\}$. Consider two deprivation matrices C and D defined as follows: the first k rows of C consist of $x^1, \cdots, x^k$, and each of the remaining rows of C is the zero vector, and the first k rows of D consist of $y^1, \cdots, y^k$, and each of the remaining rows of D is the zero vector. Note that $(\forall j \in M : |\{j : x^p_j = 1, p = 1, \cdots, k\}| = |\{j : y^p_j = 1, p = 1, \cdots, k\}|)$. It then follows that, for each $j \in M$, there is a permutation σ^j over N such that $c_{ij} \geq d_{\sigma^j(i)j}$ for all $i \in N$. Then, $C \succ^{dom} D$ follows easily. Let $i \in N$ and $j \in M$ be such that $c_{ij} = 1$. Since $\forall j \in M : |\{j : x^p_j = 1, p = 1, \cdots, k\}| = |\{j : y^p_j = 1, p = 1, \cdots, k\}|$, and by Anonymity, we can arrange $y^1, \cdots, y^k$ so that $d_{ij} = 1$. Consider C^1, which is obtained from C by changing its ij-th element from 1 to 0 while keeping all other elements of C unchanged, and D^1, which is obtained from D by changing its ij-th element from 1 to 0 while keeping all other elements of D unchanged. By Additivity (II), we then obtain $C^1 \succ^{dom} D^1$. By repeating the above argument and procedure $\sum_{j\in M} |\{j : x^p_j = 1, p = 1, \cdots, k\}| - 1 = s - 1$ times, we finally arrive at the conclusion that $C^s \succ^{dom} D^s$ where C^s and D^s are both the zero matrix, a contradiction. Therefore, (6) holds.

Then, there exist $\alpha_1, \cdots, \alpha_m$, and for all $c_{i\bullet}, d_{i\bullet} \in \{0,1\}^m$, $\varphi(c_{i\bullet}) \geq \varphi(d_{i\bullet}) \Leftrightarrow \sum_{j=1}^m \alpha_j c_{ij} \geq \sum_{j=1}^m \alpha_j d_{ij}$ (see Kraft, Pratt and Seidenberg (1959), or Fishburn (1996)). Since $\varphi(c_{i\bullet})$ is increasing in each of its argument, $\alpha_j > 0$ for all $j \in M$. Let $\omega_j = \alpha_j / \sum_{k=1}^m \alpha_k$. Then, $\omega_1 > 0, \cdots, \omega_m > 0$ and $\omega_1 + \cdots + \omega_m = 1$. Since φ is a function and given that $\varphi(c_{i\bullet}) \geq \varphi(d_{i\bullet}) \Leftrightarrow \sum_{j=1}^m \omega_j c_{ij} \geq \sum_{j=1}^m \omega_j d_{ij}$ for all $c_{i\bullet}, d_{i\bullet} \in \{0,1\}^m$, there exists an increasing function $g : [0,1] \to [0,\infty)$ such that $\varphi(c_{i\bullet}) = g(\sum_{j=1}^m \omega_j c_{ij})$. Noting that $\varphi(0_{i\bullet}) = 0$ and $\varphi(1_{i\bullet}) = 1/n$, we then have $g(0) = 0$ and $g(1) = 1/n$. $\square$

Therefore, Additivity (II) together with the axioms figured in Proposition 1 guarantees the ϕ function to be additive. The requirement that n is large relative to m is not stringent in practice. The reason for such requirement is implicit in the proof of (7) where we need a sufficient number of individuals to construct the corresponding deprivation matrices C and D.

3.3. **characterization of the second class of measures.** In this subsection, we study the structure of the second class of measures. In particular, if we replace Additivity (I) or Additivity (II) in either Propositions 2 or 3 by Strong Additivity (I), we obtain the second class of measures.

Proposition 4. *A deprivation measure h satisfies Normalization, Monotonicity, Anonymity, and Strong Additivity (I) if and only if there exist an increasing function $\xi : \{0, 1, \cdots, n\}^m \to [0, \infty)$, with $\xi(0, \cdots, 0) = 0$, $\xi(1, \cdots, 1) = 1$, and, for all $C = (c_{ij})_{n\times m} \in \mathcal{D}$, $h(C) = \xi(\sum_{i\in N} c_{i1}, \cdots, \sum_{i\in N} c_{ij}, \cdots, \sum_{i\in N} c_{im})$.*

Proof. It can be checked that, if there exist an increasing function $\xi : \{0, 1, \cdots, n\}^m \to [0, \infty)$ with $\xi(0, \cdots, 0) = 0$ and $\xi(1, \cdots, 1) = 1$ such that, and for all $C = (c_{ij})_{n\times m} \in \mathcal{D}$, $h(C) = \xi(\sum_{i\in N} c_{i1}, \cdots, \sum_{i\in N} c_{ij}, \cdots, \sum_{i\in N} c_{im})$, then this deprivation measure h satisfies Normalization, Monotonicity, Anonymity, and Strong Additivity (I). Therefore, in what follows, we prove that if a deprivation measure h satisfies Normalization, Monotonicity, Anonymity, and Strong Additivity (I), then there exist an increasing function $\xi : \{0, 1, \cdots, n\}^m \to [0, \infty)$ with $\xi(0, \cdots, 0) = 0$ and $\xi(1, \cdots, 1) = 1$ such that, and for all $C = (c_{ij})_{n\times m} \in \mathcal{D}$, $h(C) = \xi(\sum_{i\in N} c_{i1}, \cdots, \sum_{i\in N} c_{ij}, \cdots, \sum_{i\in N} c_{im})$.

Let h be a deprivation satisfying Normalization, Monotonicity, Anonymity, and Strong Additivity (I). We first show that,

$$\text{(8) if } \forall C, D \in \mathcal{D}, [\forall j \in M, |\{j : c_{ij} = 1, i \in N\}| = \{j : d_{ij} = 1\}|], \text{ then } h(C) = h(D).$$

Suppose to the contrary that (8) does not hold. Then, for some $C, D \in \mathcal{D}$, we have $[\forall j \in M, |\{j : c_{ij} = 1, i \in N\}| = \{j : d_{ij} = 1\}|]$ and $h(C) \neq h(D)$. Without loss of generality, let $h(C) > h(D)$. Since $[\forall j \in M, |\{j : c_{ij} = 1, i \in N\}| = \{j : d_{ij} = 1\}|]$, by Anonymity, we can arrange the rows of D so that, for some $i \in N$ and some $j \in M$, $c_{ij} = d_{ij} = 1$. Consider C^1 and D^1 defined as follows: C and C^1 are ij-variant and D and D^1 are ij-variant with $c'_{ij} = d'_{ij} = 0$. Then, by Strong Additivty (I), we have $h(C^1) > h(D^1)$. Note that $[\forall j \in M, |\{j : c^1_{ij} = 1, i \in N\}| = \{j : d^1_{ij} = 1\}|]$. By the repeated use of the above argument, we have $h(C^p) > h(D^p)$ where both C^p and D^p are the zero matrix, a contradiction. Therefore, (8) holds.

Since h is a function, we can define a function $\xi : \{0, 1, \cdots, n\}^m \to [0, \infty)$ so that for all $C \in \mathcal{D}$, $h(C) = \xi(\sum_{i\in N} c_{i1}, \cdots, \sum_{i\in N} c_{ij}, \cdots, \sum_{i\in N} c_{im})$. By Monotonicity, ξ is increasing. By Normalization, $\xi(0, \cdots, 0) = 0$ and $\xi(1, \cdots, 1) = 1/n$. □

To a certain degree, the result of Proposition 4 is not surprising once we realize the nature of Strong Additivity (I): it essentially says that, in measuring the overall deprivation of a society, if some individuals are deprived along a given dimension, it does not matter who is deprived as long as the same number of individuals are deprived along this dimension.

If we are willing to go one step further in requiring the overall deprivation measure satisfy Independence, from Propositions 1 and 4, the following result can be easily obtained.

Proposition 5. *A deprivation measure h satisfies Normalization, Monotonicity, Anonymity, Independence and Strong Additivity (I) if and only if there exist constants $\omega_1 > 0, \cdots, \omega_m > 0$ with $\omega_1 + \cdots + \omega_m = 1$ such that, and for all $C = (c_{ij})_{n\times m} \in \mathcal{D},\ h(C) = \sum_{i\in N}\sum_{j=1}^{m} \omega_j c_{ij}$.*

The class of measures characterized in Proposition 5 has also been obtained in Bossert, Chakravarty, and D'Ambrosio (2013) in a different setting where they deal with a richer domain by including variable societies.

Before concluding this section it may be useful to comment on the difference between the class of deprivation measures discussed in the previous section (Section 3.2) and those discussed in this section. As we noted in Section 1, the measures in the first class (following Dutta, Pattanaik and Xu (2003), we call them row-first procedures) proceed through two intutive stages: first, for every individual, i, an aggregation rule is used to aggregate the entries in the $i-$th row of the deprivation matrix to reach a meassure of i's overall deprivation, and, next, another aggregation rule is used to aggregate all the individuals' overall deprivation levels to arrive at the society's overall deprivation index. The measures in the second class (following Dutta, Pattanaik, and Xu (2003), we call them column-first procedures) proceed in a very different fashion: first, for every attrbute f_j, an aggregation rule is used aggregate the entries in the $j-$th column of the deprivation matrix to reach what may be thought of as the society's deprivation in terms of attribute f_j, and, next, an aggregation rule is used to aggregate the society's deprivation in terms of the different attributes to reach an index of the society's overall deprivation. In a framework where each individual's deprivation is cardinally measurable along the interval [0, 1], Dutta, Pattanaik and Xu (2003) have shown that, when the aggregation rule in the first stage of the row-first procedure is the same as the aggregation rule in the second stage of the column-first procedure and the aggregation rule in the second stage in the row-first procedure is the same as the aggregation rule in the first stage of the column-first procedure, the two procedures would not yield identical rankings of deprivation matrices unless the aggregation rules used in the different stages happen to be linear. Our framework is different from that of Dutta, Pattanaik and Xu (2003), and, therefore, their result cannot be exactly and directly transferred to our framework. The following example, however, illustrates that, in our framework also, there can be difference between the rankings of deprivation matrices, which are yielded by row-first and column-first procedures related to each other in a similar fashion. Consider the following two 2×3 deprivation matrices:

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Consider the class of row-first procedures figuring in Proposition 3, and let h be member of this class, such that, for all $E = (e_{ij})_{n\times m} \in \mathcal{D}, h(E) = \sum_{i\in N} g(\sum_{j=1}^{m}\omega_j e_{ij}) \equiv$

$\sum_{i\in N}(\sum_{j=1}^{m}\omega_j e_{ij})^2$.Then we have $h(C) = (\omega_1+\omega_2)^2$ and $h(D) = \omega_1^2+\omega_2^2$. Now consider a member, h', of the class of column-first procedures figuring in Proposition 4, such that, for all $E = (e_{ij})_{n\times m} \in \mathcal{D}$,$h'(E)$ =$\xi(\sum_{i\in N} e_{i1},\cdots,\sum_{i\in N} e_{ij},\cdots,\sum_{i\in N} e_{im})$ $\equiv [w_1([\sum_{i\in N} e_{i1}]/n)^2 + ... + w_m([\sum_{i\in N} e_{im}]/n)^2]$, where the weights $w_1,...,w_m$ are the same as in the specification of h. Then we have $h'(C) = \omega_1^2+\omega_2^2 = h'(D)$.

4. Conclusion

This paper has studied two classes of deprivation measures in a multi-attribute framework where the data are ordinal and every entry in the deprivation matrix is either 0 or 1. This differs from the framework of many other contributions in the literature, which assume an individual's deprivation along every dimension to be cardinally measurable along a real interval. The analysis in this paper and the analysis based on cardinal measurement of an individual's deprivation along each dimension can both be regarded as preliminary steps towards a much more general analysis, which would make use of cardinal measurement of an individual's deprivation along the dimensions that allow cardinal measurement, while simultaneously using ordinal information for the attributes for which cardinal measures are not available at the present state of our knowledge.

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